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ABSTRACT

Seven papers presented at a research conference on number and measurement are presented in this volume. The first paper provides an overview of research concerning number and measurement, and suggests directions for future research. The second paper discusses the relationships between measurement and number concepts, and psychological and instructional issues related to transfer. Two papers are devoted to synthesizing and analyzing research on measurement, and the delineation of questions about which research is needed. Two papers concern fractions; the first of these analyzes the foundations of the rational numbers from mathematical, cognitive, and instructional points of view, while the second reviews and synthesizes educational research related to fractions. The final paper concerns children's development of cardinal and ordinal number concepts. (SD)

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NUMBER

AND

MEASUREMENT

Papers from a Research Workshop

Sponsored by The Georgia Center
for the Study of Learning and
Teaching Mathematics
and the
Department of Mathematics Education
University of Georgia
Athens, Georgia

Richard A. Lesh, Editor
David A. Bradbard, Technical Editor

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MATHEMATICS EDUCATION REPORTS

The Mathematics Education Reports series makes available recent analyses and syntheses of research and development efforts in mathematics education. We are pleased to make available as part of this series the papers from the Workshop on Number and Measurement Concepts sponsored by the Georgia Center for the Study of Learning and Teaching Mathematics.

Other Mathematics Education Reports make available information concerning mathematics education documents analyzed at the ERIC Information Analysis Center for Science, Mathematics, and Environmental Education. These reports fall into three broad categories. Research reviews summarize and analyze recent research in specific areas of mathematics education. Resource guides identify and analyze materials and references for use by mathematics teachers at all levels. Special bibliographies announce the availability of documents and review the literature in selected interest areas of mathematics education. Reports in each of these categories may also be targeted for specific sub-populations of the mathematics education community.

Priorities for the development of future Mathematics Education Reports are established by the advisory board of the Center, in cooperation with the National Council of Teachers of Mathematics, the Special Interest Group for Research in Mathematics Education, and other professional groups in mathematics education. Individual comments on past Reports and suggestions for future Reports are always welcomed by the ERIC/SMEAC Center.

Jon L. Higgins
Associate Director

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Acknowledgements and Overview

The Georgia Center for the Study of Learning and Teaching Mathematics (GCSLTM) was started July 1, 1975, through a founding grant from the National Science Foundation. Various activities preceded the founding of the GCSLTM. The most significant was a conference held at Columbia University in October of 1970 on Piagetian Cognitive-Development and Mathematical Education. This conference was directed by the late Myron F. Rosskopf and jointly sponsored by the National Council of Teachers of Mathematics and the Department of Mathematical Education, Teachers College, Columbia University with a grant from the National Science Foundation. Following the October 1970 Conference, Professor Rosskopf spent the winter and spring quarters of 1971 as a visiting professor of Mathematics Education at the University of Georgia. During these two quarters, the editorial work was accomplished on the proceedings of the October conference and a Letter of Intent was filed in February of 1971 with the National Science Foundation to create a Center for Mathematical Education Research and Innovation. Professor Rosskopf's illness and untimely death made it impossible for him to develop the ideas contained in that Letter.

After much discussion among faculty in the Department of Mathematics Education at the University of Georgia, it was clear that a center devoted to the study of mathematics education ought to attack a broader range of problems than was stated in the Letter of Intent. As a result of these discussions, three areas of study were identified as being of primary interest in the initial year of the Georgia Center for the Study of Learning and Teaching Mathematics--Teaching Strategies, Concept Development, and Problem Solving. Thomas J. Cooney assumed directorship of the Teaching Strategies Project, Leslie P. Steffe the Concept Development Project, and Larry L. Hatfield the Problem Solving Project.

The GCSLTM is intended to be a long-term operation with the broad goal of improving mathematics education in elementary and secondary schools. To be effective, it was felt that the Center would have to include mathematics educators with interests commensurate with those of the project areas. Alternative organizational patterns were available--resident scholars, institutional consortia, or individual consortia. The latter organizational pattern was chosen because it was felt maximum participation would be then possible. In order to operationalize a concept of a consortia of individuals, five research workshops were held during the spring of 1975 at the University of Georgia. These workshops were (ordered by dates held) Teaching Strategies, Number and Measurement Concepts, Space and Geometry Concepts, Models for Learning Mathematics, and Problem Solving. Papers were commissioned for each workshop. It was necessary to commission papers for two reasons. First, current analyses and syntheses of the knowledge in the particular areas chosen for investigation were needed. Second, a catalyst for further research and development activities were needed--major problems had to be identified in the project areas on which work was needed.

Twelve working groups have emerged from these workshops, three in Teaching Strategies, five in Concept Development, and four in Problem Solving. The three working groups in Teaching Strategies are: Differential Effects of Varying Teaching Strategies, John Dossey, Coordinator; Development of Protocol Materials to Depict Moves and Strategies, Kenneth Retzer, Coordinator; and Investigation of Certain Teacher Behavior That May Be Associated with Effective Teaching, Thomas J. Cooney, Coordinator. The five working groups in Concept Development are: Measurement Concepts, Thomas Romberg, Coordinator; Rational Number Concepts, Thomas Kieren, Coordinator; Cardinal and Ordinal Number Concepts, Leslie P. Steffe, Coordinator; Space and Geometry Concepts, Richard Lesh, Coordinator; and Models for Learning Mathematics, William Geeslin, Coordinator. The four working groups in Problem Solving are: Instruction in the Use of Key Organizers (Single Heuristics), Frank Lester, Coordinator; Instruction Organized to use Heuristics in Combinations, Phillip Smith, Coordinator; Instruction in Problem Solving Strategies, Douglas Grouws, Coordinator; and Task Variables for Problem Solving Research, Gerald Kulm, Coordinator. The twelve working groups are working as units somewhat independently of one another. As research and development emerges from working groups, it is envisioned that some working groups will merge naturally.

The publication program of the Center is of central importance to Center activities. Research and development monographs and school monographs will be issued, when appropriate, by each working group. The school monographs will be written in nontechnical language and are to be aimed at teacher educators and school personnel. Reports of single studies may be also published as technical reports.

All of the above plans and aspirations would not be possible if it were not for the existence of professional mathematics educators with the expertise in and commitment to research and development in mathematics education. The professional commitment of mathematics educators to the betterment of mathematics education in the schools has been vastly underestimated. In fact, the basic premise on which the GCSLTM is predicated is that there are a significant number of professional mathematics educators with a great deal of individual commitment to creative scholarship. There is no attempt on the part of the Center to buy this scholarship--only to stimulate it and provide a setting in which it can flourish.

The Center administration wishes to thank the individuals who wrote the excellent papers for the workshops, the participants who made the workshops possible, and the National Science Foundation for supporting financially the first year of Center operation. Various individuals have provided valuable assistance in preparing the papers given at the workshops for publication. Mr. David Bradbard provided technical editorship; Mrs. Julie Wetherbee, Mrs. Elizabeth Platt, Mrs. Kay Abney, and Mrs. Cheryl Hirstein, proved to be able typists; and Mr. Robert Petty drafted the figures. Mrs. Julie Wetherbee also provided expertise in the daily operation of the Center during its first year. One can only feel grateful for the existence of such capable and hardworking people.

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Directions for Research Concerning Number and Measurement Concepts

Richard Lesh

Northwestern University

In the Spring quarter of 1975, under a grant from the National Science Foundation, the Georgia Center for the Study of Learning and Teaching in Mathematics (GCSLTM) sponsored a series of five research workshops in mathematics education. The workshops were each three to four days in length and the topics considered were: (a) teaching strategies in mathematics, (b) number and measurement concepts, (c) space and geometry concepts, (d) models for learning mathematical concepts, and (e) problem solving. This monograph contains the papers that were presented at the number and measurement workshop.

The purpose of these remarks is to mention some of the ideas, opinions, and unifying themes that arose during discussion sessions at the workshop but which are not explicitly mentioned in the papers included in this monograph. The remarks are separated into two parts. The first has to do with the general milieu in which research is conducted. The second has to do with specific trends concerning research about the acquisition of number and measurement concepts.

The General Climate for Future Research

Before mentioning some specific themes concerning directions for future research on the acquisition of number concepts and measurement concepts, it seems appropriate to mention some general trends that affect the general climate in which this research is conducted.

The Scarcity of Funds for Mathematics Education Research

Because of their complexity, most of the really important problems in mathematics education require long-term commitments and coordinated research efforts from many individuals. Yet, coordinated research efforts usually occur within large funded projects. A basic problem which confronts the profession of mathematics education is to operationalize coordinated research efforts with minimal funding from federal or private agencies.

Actually, a large portion of the research that is done in mathematics education is conducted either by (a) doctoral students as dissertations, or (b) individuals working at institutions where there are few colleagues to "bounce ideas off of," and where there are few funded projects. So, some questions that should be considered are: "How can these research efforts be guided and coordinated so that steps can be taken toward finding answers to some of the more complex and important problems in mathematics education?" "When minimum funds are available, how can we avoid having unqualified people making major decisions about directions for future research?" "How can projects be funded without being forced to work under artificial accountability procedures that doom them to failure before they begin?"

Several aspects of the workshops sponsored by the GCSLTM were particularly pleasing to participants because they offered alternative solutions to several of the problems presented by shortages in research funds. Rather than deciding ahead of time what research questions ought to be investigated and then parceling out subprojects to individual researchers, or attempting to buy research through salaries paid to individual researchers, the GCSLTM identified several research areas that are particularly important and then brought together people who already were doing research in these areas. In this way, people who are most knowledgeable about an area can decide for themselves the directions that their own future research efforts should take. Furthermore, the project can capitalize on what has already been done rather than attempting to "reinvent the wheel" in a given research area.

During times when funds are scarce, more thought needs to be given to ways of capitalizing on resources that are already available. In mathematics education, this means considering ways to amplify the effectiveness of people who do research on a regular basis--often without outside research funding. There is a big difference between: (a) spending money to buy research, and (b) spending money to promote cooperation and facilitate communication among people who are already actively involved in research--often without any form of monetary support. The Georgia workshops elected to focus on the latter type of objectives. In this way, it was possible to stimulate research in an area by providing encouragement for more people to work in that area, and it was possible to influence the direction research will take without stifling the initiative of researchers who are most knowledgeable and most committed to research in the area. Furthermore, the final directions that research efforts will take is determined cooperatively by (a) individual researchers, (b) a group of researchers acting collectively, and (c) the funding agency--with none of these forces dominating the others.

When experienced researchers work together for several days to isolate individual research projects, the chances increase that the projects that evolve will be more basic, more to the heart of the issues, and consequently more important. Furthermore, because many individuals will have input into the planning, and because many individuals will have an opportunity to coordinate their research efforts, more complex issues can be investigated. The optimal time to establish connections among individual research efforts is while project plans are in the formative stages of development--not a year or two after projects have been completed and reports finally appear in journals or at conferences.

Communication Problems

One of the most disconcerting facts about mathematics education research is that, for most topics like the ones considered at the number and measurement workshop, a great deal of information has been accumulated and yet very little is known. Part of the difficulty stems from the apparent unwillingness of mathematics educators to take the time to seriously investigate the psychological literature available to determine its applicability to the learning of mathematics. While such investigation is difficult, time consuming, and at times unrewarding, it can contribute immeasurably in two ways. First, psychology offers one point of departure for understanding mathematical learning. It may offer structure to problems heretofore unstructured and methods of research not commonly used in mathematics education. Second, the psychological literature itself can be evaluated concerning its applicability to the problems confronting mathematics education. For example, terminology of a mathematical nature--"topology," "quantity," "number"--have meanings in mathematics different than meanings in psychology. Consequently, deciphering the mathematical meaning of mathematical-like terminology psychologists use is crucial for a correct interpretation of their work as it relates to the acquisition of mathematical concepts.

For example, at the workshop on number and measurement concepts, it was found that Brainerd used mathematical language in a way that did not correspond to the usage of mathematicians. While this does not lessen the importance of Brainerd's excellent work, mathematicians must be alert to the differences in terminological usage and not generalize the research results unwarrantedly to their own conceptual referents.

Psychological theory other than Brainerd's (Piaget's) is available concerning the development of primitive number concepts in children. But Brainerd apparently does not ascribe to Piaget's developmental theories concerning number. The disagreement appears to be based more on assumptions about the mathematical foundation of number than on cognition. Consequently, it is critical that the mathematics educator look carefully at the psychologists' mathematics in order to understand his psychology. A few days of discussion with well-informed colleagues can do a great deal to broaden the knowledge base for individual researchers. Some of the information that evolves from such personal interaction can evolve in no other way. The research workshop on number and measurement provided a forum for discussion of terminological difference and heightened the participants' awareness of such differences.

The research workshop also provided a rare setting in which to share inaccessible references and results of ongoing work. Accessibility of information can be just as much a roadblock to effective communication as terminological problems. Ideas that are published only in obscure or unobtainable journals (e.g., Soviet research) were shared, and opinions were traded that were based on data from ongoing research projects, preliminary pilot studies, or studies that were partial failures. Furthermore, participants become familiar with projects that were unpublished or projects that were in the formative stages of development. Participants also came in contact with relevant research that had been overlooked during months of preliminary library work (e.g., a recent book by Bryant dealing with the acquisition of early number concepts titled Perception and Understanding in Young Children (1974) was unfamiliar to many participants). Furthermore, because the participants came from mathematics, mathematics education, and psychology, communication problems were, in some cases, ironed out "on the spot."

Certainly, if progress is ever to be made on many of the most complex issues that are most important to mathematics educators, groups of researchers will have to develop better bases of communication so that individuals can profit by (and build on) the work of others. At the present time, coordinating research efforts that are already being made (or that have already been made) is at least as important as attempting to generate more information. The next section of this paper will discuss some "levers" that could be used to influence the direction research will take in particular areas. Again the emphasis will be away from buying research and toward stimulating, focusing, and coordinating research efforts that are already being made.

The Tight Job Market

Compared with some "years of plenty" in the recent past, the job market will probably remain tight in mathematics education as well as in mathematics. Among the problems that arise in conjunction with a tight job market, the following seem relevant to mention here.

1. The job market is even tighter in mathematics than in mathematics education, and it has been tight for a longer period of time. Consequently, many research mathematicians are taking jobs in smaller and less well-known institutions where they are often confronted by educational problems rather than mathematical problems. Sometimes, this means that an excellent mathematical scholar may direct his efforts towards educational issues. It is appropriate to mention that such people can benefit greatly from mathematics education research conferences. Such conferences are quite common in mathematics, but (except for rare exceptions such as the Georgia workshops) are almost nonexistent in mathematics education.[1]

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[1] Research sessions at meetings of the National Council of Teachers of Mathematics or the American Education Research Association simply do not serve the same function as research workshops and conferences. In a research conference, it is usually assumed that the participants are already knowledgeable about the subject of the conference, and that their interests are in coordinating research, in forming generalizations, in discussing

2. Many well-trained mathematics educators who would like to become involved in research projects are now working at smaller institutions where (a) a tradition of research has not been established, (b) research facilities (computers, consultants, etc.) are not available, (c) colleagues are not available, and (d) graduate assistants and doctoral students are not available. Nonetheless, mathematics educators in this category have produced a significant quantity of research in recent years. Part of the reason for this fact is that, because of the tight job market, many institutions are making demands on their faculty that would have been unrealistic a few years ago. A "publish or perish" criterion is now being applied even in nonresearch institutions which provide little other than threats (concerning salary, promotion, and tenure) to encourage research activities on the part of their faculty.

3. For mathematics educators who work in mathematics departments, "to publish" is sometimes interpreted as "to publish research," and this is so in spite of the fact that mathematics educators are frequently saddled with excessive teaching (supervising and advising) loads and little monetary support for scholarly activities (e.g., travel funds to attend conferences, typing costs, computer time costs, publication costs, etc.). Consequently, even though this "rush to publish" phenomenon may be unjust, it is a fact of life for many mathematics educators. Nonetheless, rather amazingly, a significant number of mathematics educators are doing some interesting work under these highly adverse conditions.

4. Even at major institutions where research has been a tradition, the tight job market and the scarcity of research funds are having serious effects. Fewer funds are available for all sorts of scholarly activities, and far fewer temporary "soft money" persons are available. Furthermore, teaching loads are increasing as hard money positions evaporate in the wake of declining enrollments in schools of education.

5. Mathematics education professors are becoming far less mobile than in the past. That is, once having reached the rank of associate professor or full professor, it is very difficult to get a job at another institution. Almost all new faculty positions are at the assistant professor level. So, most institutions are faced with a stable and permanent faculty with far fewer opportunities for an influx of new people with new ideas. Consequently, it has become important for major institutions to provide opportunities for its faculty to confront new ideas and new colleagues,

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data, or in determining directions for future research. The goal is not simply to disseminate conclusions to prospective consumers of research information.

Certainly, one objective of mathematics education research should be to indicate implications for consumers, but this should not be the only objective. In fact, because of the complexity of the issues that arise in mathematics education, few isolated studies will ever produce results that yield definitive generalizations about most of the really important issues that concern teachers. If overtly simplistic answers are to be avoided, most issues will require intensive study by various inidviduals.

and for faculty members to continually reeducate themselves to "keep fresh" and remain on the forefront of knowledge in their fields. Unfortunately, however, sabbaticals and other policies that used to be aimed at providing for the educational needs of professors, are becoming things of the past in most institutions. Furthermore, this is happening at a time when university salaries are so low that few professors can afford to provide for their own released time, travel funds, publication costs, etc. Nonetheless, funding agencies seem to have given little thought to ways of using these needs as "levers" to influence and coordinate the research activities of mathematics educators or their doctoral student advisees.

Of the main types of individuals mentioned above (i.e., mathematicians who become interested in educational issues, psychologists who become interested in the acquisition of mathematical concepts, mathematics educators at smaller institutions, and mathematics educators at major universities), all are in need of research workshops, conferences, and seminars where they can reeducate themselves by exchanging ideas with colleagues from other disciplines and other institutions, and where they can develop better bases of communication about important topics.[2] Consequently, these "needs" can be used as "levers" to coordinate and focus research concerning topics that are judged to be particularly important.

Graduate students in mathematics education could also benefit greatly from intra-institutional cooperative research efforts. As mathematics education departments decrease in size at most major institutions, and as the influx of new (short-term or soft money) faculty decreases, it will become more difficult for Ph.D. students to become involved in a variety of different types of projects. Furthermore, this will happen at a time when the tight job market will force serious doctoral students to develop stronger credentials at the pre-Ph.D. level. Consequently, serious doctoral students will give greater consideration to becoming involved in pre-dissertation research and development projects. Pre-dissertation research has been typical for years in mathematics and in psychology, and it seems likely that professional activities of a variety of different types will become increasingly important at the pre-Ph.D. level in mathematics education. But again, this will mean that doctoral students and their advisers must have access to intra-institutional cooperative research and development projects.

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[2] General information centers, like ERIC do not serve the functions indicated here. Smaller information centers focused on specific types of topics are needed. For smaller communication centers, the goal is to be able to bring individuals together who want to do research in particular areas.

Another type of service that could be very helpful to college and university educators is to provide more and different types of publication outlets. For example, the Journal for Research in Mathematics Education (JRME) is one of the few research outlets for mathematics educators, and JRME only comes out four times per year with about six articles per issue. The flood of articles submitted has produced time lags of several years between the time that many articles are submitted and the time of publication.

The participants at the number and measurement workshop were pleased that during the year following the workshop the GCSLTM intends to publish a "number and measurement" research monograph including research reports and related papers written by Center affiliates. Furthermore, monographs (analogous to this one) are also being published for each of the other workshops. Publication in the monographs of the Center does not preclude submitting the article to other journals. In fact, publication in other journals is encouraged.

The potential productivity of the Georgia Center is encouraging. By using a relatively small grant from the National Science Foundation, and by capitalizing on the many resources that are already available in mathematics education, the Georgia Center will have (a) held five research workshops in the Spring of 1975, (b) produced five monographs from the 1975 workshops with each monograph providing directions for research concerning those topics that were selected for special attention, (c) held eight two-day workshops (e.g., number and measurement concepts, space and geometry concepts, teaching strategies, and problem solving) prior to the 1976 annual NCTM meeting in Atlanta, (d) produced research monographs reporting work growing out of the 1975 workshops, and (e) throughout the 1975-76 academic year, attempted to facilitate communication and promote cooperation among project associates.

Using grant money to focus and coordinate resources that are already available is a simple, sensible, and powerful idea. Speaking for many participants who benefited greatly from the number and measurement workshop (or from one of the other workshops), I hope that the Georgia Center will continue its work in future years, and that similar projects may be created at other institutions. There is no reason why the basic "modus operandi" of the Georgia Center cannot also be applied to curriculum development projects as well as research oriented projects.

One idea that is not an aspect of the Georgia Center, but which could be considered in future years (or by other projects), would be to provide some small amounts of money for unusual research experiences incurred by project associates. For example, in most major research institutions, there are research funds available for graduate students or for junior faculty members which can be used for minor expenses (e.g., \$100-\$500) to help defray such expenses as computer time, materials needed as research equipment, etc. Furthermore, free consulting services are often available involving statistical design or computer programming. But, such services are usually not available to most mathematics educators--especially those working at smaller colleges or universities.

In major research institutions, the idea is to use the general reputation of the university, or the reputations of senior faculty members, to create research funds that can be used to encourage younger researchers. Research funds of this type can often be used very productively because: (a) money is only committed after a project is well beyond the preliminary planning stages, (b) only a fraction of the cost of the project is covered by the small grants, and (c) because senior faculty members have an opportunity to influence research work of younger researchers. In other words, a minimum investment of funds often produces rather large results because the goal is not to buy research but rather to encourage and improve the research efforts that are already being made.

Attitudes Toward Research

One of the most important factors influencing number and measurement research is the general attitude that mathematics educators have toward research. At cocktail parties it will continue to be acceptable (in fact fashionable) to claim to lack a "mathematical mind" and to have little ability in arithmetic or interest in mathematics. But it is hoped that at mathematics education conferences, it will soon be unacceptable (in fact not fashionable) to claim to have little interest or knowledge about "research."

Mathematics educators are justifiably critical about the quality of research in mathematics education. But, some mathematics educators use complaints about past research as an argument against doing research, having research meetings at professional meetings, or having a professional journal for research in mathematics education. But to criticize poor research is not the same as criticizing research in general---especially if the criticism minimizes the chance that mathematics educators will ever find adequate solutions to many of their most important problems. Research is the act of trying to find useful information for school mathematics.

The nature of mathematics education.

Research must not be mindless data gathering followed by nit-picking analyses. Good research involves (a) identifying an important issue, (b) formulating basic (answerable) questions related to the issue, (c) determining answers that are useful in a variety of situations, and (d) communicating the results and conclusions in a meaningful way to other mathematics educators. None of these four aspects of research seem objectionable. What is it that mathematics educators find objectionable about past research?

The real problems with research are that many of the most important issues have been neglected; the questions we have asked and the answers we have obtained are often superficial or overtly simplistic. Many studies are impossible to replicate, and so their results are not useful to other mathematics educators in other situations. We have not developed bases of communication so that teachers and other researchers can use the results that are obtained. Therefore, we should attack these problems--not research.

Because of the complexity of most of the important issues in mathematics education, asking an important question seems to be almost irreconcilable with asking an answerable question. However, this is only true if one must restrict oneself to asking single isolated questions. The problem is far less acute if groups of researchers work together on sets of related questions, or if individual researchers build on the work of others over an extended period of time. This means that most of the problems mentioned above are directly related to problems of communication among mathematics educators.

Who does research in mathematics education? Some people seem to believe that researchers must necessarily be "ivory tower" individuals who know little about the "real world" of teaching and instructional development. However, in mathematics education, many of our best researchers are also among those individuals who have the greatest contact with children in the schools. In fact, in mathematics education, research and curriculum development seem to go hand-in-hand. That is, it is difficult to continue to come up with good research questions over a long period of time if one avoids teaching situations and curriculum development projects; it is difficult to be a creative writer of instructional materials if one avoids asking (and seeking answers to) basic questions about teaching and learning. So, in the opinion of the author, good research and good curriculum development must be closely related. For example, participants at the Georgia research workshops representing curriculum projects far outnumbered participants representing research projects.

The influence of past research on classroom practice. Some mathematics educators who claim to be good teachers insist that research has had little influence on their teaching. This is naive. Every time a teacher teaches, every time a set of instructional materials is developed, the teacher or authors operated on some basic assumptions (perhaps unarticulated) about teaching and learning. Unfortunately, however, the assumptions are seldom more than "rules of thumb" that are often inappropriate for some children or for some teachers in some situations. "Use concrete activities before abstract," "create intuitive understanding before formalization," or "use discovery rather than reception methods;"--each of these slogans is appropriate in some situations and clearly inappropriate in others. Yet, the range of appropriateness of such slogans has seldom been investigated. Consequently, teachers are forced to operate on the basis of a "theory" consisting of nothing more than a hodge-podge of unorganized, unexamined, and (most always) overtly simplistic set of slogans.

It is naive to claim that a teacher teaches without at least some untested assumptions or that a textbook is written without the same. The only question is whether the assumptions are good ones or poor ones. A good theory is one that can be communicated to others, and that avoids obvious errors, oversimplifications, and inconsistencies. In a paper given at the workshop on space and geometry (Lesh, 1975), I commented, and repeat here that the

> cyclic history of curriculum change (i.e., enthusiastic adoption, followed by disillusionment, followed by rejection) indicated that theory building has not really been taken seriously by mathematics educators. If mathematics education is ever going to make lasting progress on some of its most important problems, then the emphasis in research must shift away from "undirected information gathering" and toward "theory building." Theory building does not necessarily have to conjure up images of dull "ivory tower" activities that make no real difference anyway. For a beginning, theory building can simply involve a point of view that can form a basis for communication with other mathematics educators. In this way, individuals can profit by (and build on) the work of others. Also, in order to avoid obvious errors and inconsistencies, theory building should attempt to describe the range of applicability of its major principles, and should reconcile major conflicts within its point of view. When difficulties arise, a theory should be more than a point of view that is simply accepted or rejected; it should be an explanatory "model" that can (and must) be gradually modified and reorganized to deal with progressively more complex situations. (pp. 5-6)

Many of the problems with mathematics education research stem from communication problems among mathematics educators and communication problems between mathematics educators and people in other disciplines. Furthermore, these problems have been compounded because mathematics education research has been viewed as an activity that is not integrally related to curriculum development and as an activity that can take place without being accompanied by theory development.

Over a prolonged period of time, it seems likely that mathematics educators must take responsibility for their own research and theory building. Psychologists and mathematicians should not be expected to answer the questions of mathematics education. Nonetheless, it is important to remember that, in the past, many of the people who have been most influential in mathematics education have come from outside of mathematics education. So, it is productive to try to attract the psychologists and mathematicaians to take an interest in basic issues in mathematics education. One of the best ways to encourage such interests is through research workshops where psychologists, mathematicians, and educators can get together for intensive discussions over several days.

Because research is the act of trying to find useful answers to important questions, it is naive to ask whether we should have research. We will have research; the only question is whether it will be good research or poor research. If the questions are superficial or if the results are nothing more than "teaching tricks" that work only in particular situations, then the research is poor. But, if groups of people can identify important issues that can be attacked cooperatively, and if the results of individual research efforts are communicated in such a way that they are useful to other mathematics educators, then the result should be better research and more relevant applications in teaching and instructional development.

Summary and Conclusions

Throughout the previous sections, a number of factors were mentioned that effect the general milieu in which mathematics education research will be conducted in coming years. The problems that were mentioned had to do with the scarcity of research funds, communication problems among researchers, the tight job market, and general attitudes about research among mathematics educators. The solutions to these problems had to do with using research funds to focus and coordinate resources that are already available. To do this, it was suggested that the trend should be toward promoting cooperation and communication among people who are already actively involved in research rather than spending money to buy research. Some of the needs that have been created by shortages of funds and by the tight job market furnish levers that could be used to influence the direction that research will take without stifling the initiative of researchers who are most knowledgeable and most committed to those areas that are chosen to receive special attention. By encouraging groups of researchers to work together on cooperative (or at least related) projects, more important issues can be attacked and more useful results can be obtained.

The research workshops that were sponsored by the Georgia Center for Learning and Teaching Mathematics seemed to offer a means of making efficient use of resources that are already available and for facing some of the severe problems that researchers will confront in coming years. The workshops also recognized the fact that instructional development and theory building must both be integrally related to future research efforts that are made, and that coordinating existing research efforts is at least as important as attempting to generate more information.

Trends in Future Research Concerning Number and Measurement Concepts

The workshop on number and measurement was organized into three working groups: cardinal and ordinal number, rational number, and measurement.

On the first day of the workshop, it seemed that the three working groups might have little in common. But, as discussions proceeded, some common themes and some mutual interests emerged. For example, one unifying theme had to do with the function and use of concrete materials to help children develop basic concepts about rational numbers or counting numbers. It was clear that many children's misunderstandings about number concepts are closely linked to misunderstandings about the models that are used to teach the concepts, and it was clear that different models often emphasize different aspects about a particular concept. But there were many disagreements about how concrete materials can be used most effectively. For example, each of the materials in Figure 1 can be used to illustrate rational number concepts, and each emphasizes an aspect of the number 1/3. It can be argued that each of the types of materials is "good" in some ways and "not so good" in others. Some of the materials stress the "fraction" or "part of a whole" interpretation of rational numbers. Others emphasize the "ratio" or "proportion" interpretation of rationals. And, others emphasize the "ordinal" or "operator" interpretation of rationals. Still others discuss rationals as "ordered pairs" or as extensions of our numeration system. Other issues also arise. For instance, which materials are most abstract, or concrete, or complex? Which will be easiest for youngsters to use? Which materials will allow youngsters to deal most directly with the most elementary interpretations of rationals and yet not lead them to form misconceptions that will make higher order understanding more difficult (e.g., youngsters who have learned that rational numbers refer to "parts of a whole" may find difficulties when they confront three-halves). What role does familiarity play in selecting materials? Which materials will draw upon more useful intuitive notions without also conjuring up irrelevant properties? How many different types of materials should be used, and in what order should they be presented? Finally, are there any generalizations that can be made about discrete models versus continuous models, or about cardinal-versus-ordinal-versus-measurement models?

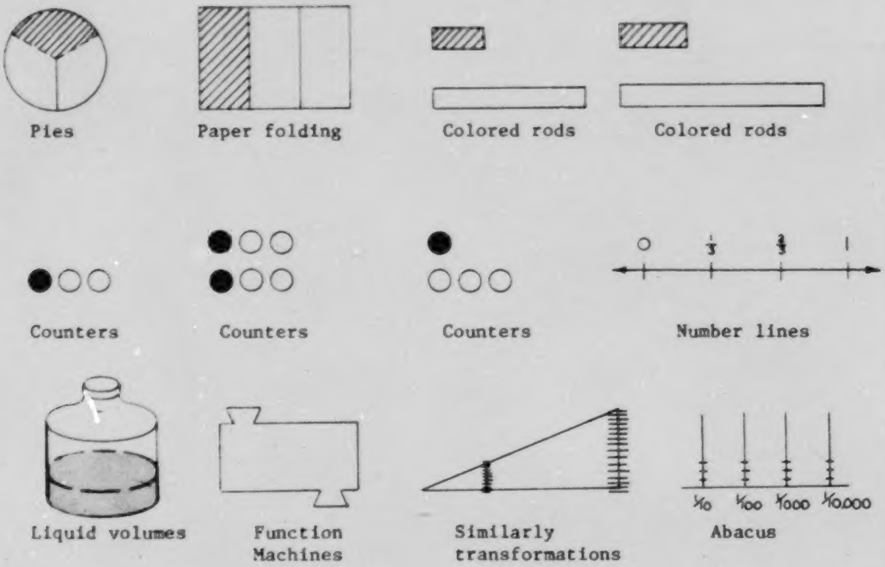

Figure 1. Illustrations of 1/3.

The cardinal and ordinal number group focused on the acquisition of early number concepts and was more self-consciously Piagetian (or at times anti-Piagetian) than the rational number group, but many of the discussions centered around similar issues. For instance, Brainerd argued that psychologists and mathematics educators have neglected ordinal number ideas, and that researchers and curriculum developers have focused almost exclusively on cardinal number concepts. Yet, Brainerd pointed out in his paper that ordinal concepts develop earlier than cardinal concepts. On the other hand, other participants pointed out that Brainerd's use of the words "cardinal number" and "ordinal number" were not entirely consistent with their usual meanings, and several expressed the belief that Brainerd's results were obtained because his studies were systematically biased in favor of (what he called) "ordinal number" concepts. For example, some of Bryant's work (1974) seems to indicate that certain "ordinal" ideas also develop at about the same time as Brainerd's "ordinal" concepts. Or, some of Steffe's work (1973) might argue that the concepts that Brainerd and Bryant have studied are not number concepts at all but rather are prenumber concepts. It could be that arguments about children's early number concepts miss the point. Perhaps the real issue is whether it is appropriate at all to analyze tasks or concepts on the basis of their underlying mathematical structure. For instance, Piaget's theory is based on the hypothesis that it is appropriate to order and equate tasks and concepts on the basis of their underlying operational structures, but an analysis of the figurative aspects of a task often seem to be equally as important as its operational aspects, and certain "information processing" variables (see Carpenter and Osborne's paper in this monograph) seem to account for a great deal of the variability among tasks.

In all of the above discussions, the word "structure" became a central idea. Yet, "structure" was used in at least three different ways corresponding to three ways that mathematical concepts can be organized. There are: (a) mathematical structures, the way the discipline (or mathematician) organizes a set of concepts; (b) instructional structures, the way the teacher, textbook, or instructional program organizes a set of concepts; and (c) cognitive structures, the way a student (or child) organizes the set of concepts.[3] Furthermore, it is easy to find a learning theorist who supports almost any conceivable connection between these three types of structures (e.g., cognitive structures determine instructional structures--Piaget; instructional structures determine cognitive structures--Gagné; disciplinary structures determine instructional structures--Ausubel). The point is that there may frequently be an important distinction between the logical development of an idea and the way students come to understand it. Unfortunately, however, for many of the ideas discussed at the number and measurement workshop, surprisingly little is known about the way children organize mathematical concepts. Yet, concerning psychological theories that seem most relevant to mathematics educators, preconceived biases about the nature of mathematical concepts appear to account for at least as much variance among theorists as a priori biases about the nature of cognition.

It is remarkable that so little is known about the nature of children's early conceptions of many mathematical ideas. This may be because the first mathematical judgements children learn to make relative to a given mathematical idea are highly specialized, closely tied to specific content, and highly restricted. But the ideas that mathematicians use as building blocks for their theories are those that are most general, those which can be used in the widest range of circumstances, and those that give rise to "nice" theories. If we try to use children's first concepts as building blocks for a theory (for example, as Grize (Beth & Piaget, 1966) has done using Piaget's grouping structures), we would find "messy" primitive concepts that do not give rise to neat, tidy, elegant theories. For this reason, mathematicians have not taken the trouble to formalize such awkward structures--especially as building blocks for a theory. In fact, it seems unlikely that mathematicians will ever take the trouble to describe most of the structures children use when they first come to master a given idea.

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[3] Perhaps there is also a fourth category which could be called psychologists' structures corresponding to the way psychologists organize the concepts.

Some of the best resources we have describing the nature of children's early number concepts have come from Piagetian studies. Nonetheless, because Piagetian research has focused on the formation and description of cognitive processes used by first-graders (i.e., concrete operational groupings) and by sixth-graders (i.e., INCR groups), children at intermediate levels of development have been neglected. Furthermore, because psychologists in general (and Piagetians in particular) have avoided mathematical ideas that are typically taught in elementary school, it is often possible to make only relatively crude inferences about how children's mathematical thinking gradually changes from concrete operational concepts to formal operational concepts. It is time for mathematics educators to apply Piagetian techniques and theory to concepts like rational numbers and negative numbers that exist at intermediate levels of development. For example, Kieren's paper represents a first step in the direction of a Piagetian analysis of the concept of rational numbers.

To examine the nature of children's conceptions concerning a given idea, it is useful to begin by sorting out various mathematical interpretations of the idea and then to devise a variety of concrete embodiments of each of these interpretations. In this way, our knowledge about mathematical structures can be used to investigate children's cognitive structures, and information about cognitive structures can be used to direct the development of instructional structures (see Figure 2).

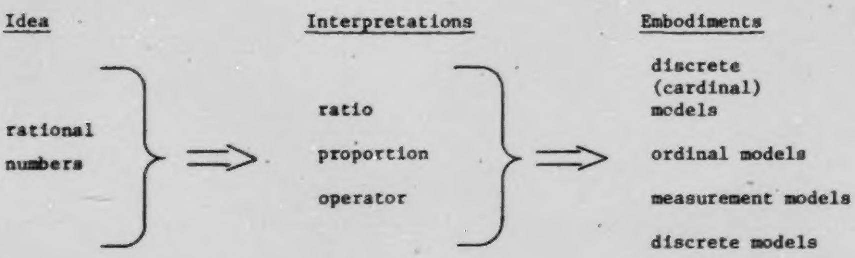

Figure 2

Steffe (1973) and others have argued that it is possible to describe children's behavior on number related tasks using well-defined systems that already exist in mathematics. That is, there may be no need to invent new and unfamiliar systems (like Piaget's groupings) to model children's thought processes concerning number concepts. Furthermore, at the space and geometry workshop, I described a method of using information about the mathematical structure of an idea to guide research concerning children's cognitive structures concerning the idea. Similarly Osborne and Carpenter, and Osborne discuss ways of using information about the mathematical structure of concepts to make decisions about instructional sequencing and to make predictions about transfer of learning.

In the discussions that took place at the number and measurement workshop, distinctions were made concerning various modes in which a given idea might be investigated. There is written symbolic mathematics; there is spoken mathematics; there is picture symbolic mathematics; there is mathematics in the context of concrete models; and there is mathematics in real world situations.

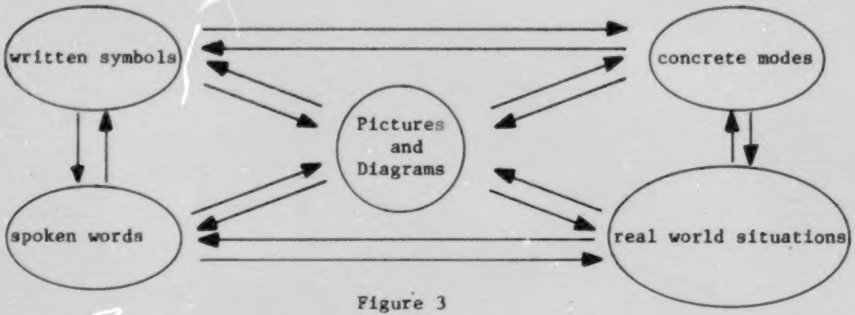

Figure 3

Children may experience difficulties within any one of the above levels (e.g., illustrate 2 + 3 = using a number line and ask for an illustration using counters); or, within a given situation, they may have difficulties translating between various interpretations of a given concept (e.g., ask a child to put out a row of counters and to label them "1," "2," "3," "4," and so forth. Then cover the first seven counters with a handkerchief, and ask how many counters are covered. The answer requires the child to use the ordinal information that is given to make a judgment about the cardinality of a set). Children may also have difficulties translating from one mode to another (e.g., from spoken words to written symbols, from concrete models to real world situations, from pictures to concrete models, etc.). Yet, many of these within-mode translation problems have been neglected by both authors and researchers. Nonetheless, information processing variables from psycholinguistic literature could furnish some useful ideas for improving mathematics instruction.

Among the many research projects that were planned by participants at the number and measurement workshop, the following four categories of studies seem relevant to mention in conjunction with what has been said in these comments.

1. Piagetian style studies were planned aimed at investigating the nature of children's primitive conceptions about rational numbers and counting numbers.

2. Studies were planned aimed at investigating the operative aspects of certain tasks. These studies would involve some of the major types of instructional materials that are typically used to illustrate rational numbers and counting number concepts.

3. Studies were planned aimed at investigating ways that measurement activities could be used to facilitate the developmental number concepts. More then 60 years ago, Dewey and McClellan (1914) wrote a book describing ways that number concepts develop out of measurement activities, and the points that they made have been reinforced by Piaget, Inhelder, and Szeminska (1964) and by Soviet psychologists like Gal'perin and Georgiev (1969). But, with the exception of a few projects (like *Developing Mathematical Processes*, 1974), very little has been done to develop materials that use measurement activities as a means of facilitating the acquisition of number concepts. Measurement activities have tended to be thought of as "applications" that can only be presented after number concepts have been learned.

4. Coordinated series of "teaching experiments" like the rational number studies recently conducted at the University of Michigan (see Payne's paper) are becoming increasingly interesting to researchers. For example, Steffe (1975) and Kantowski (1975) have both discussed the use of "teaching experiments" to investigate the acquisition of number concepts.

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The Mathematical and Psychological Foundations of Measure

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Introduction

A measure system is a complex structure of ideas. The child must learn many such systems in maturing to adulthood. Each measure system, whether denoted by a word such as length, angularity, work, time, area, energy, volume, or pressure, is based on a characterizing function. This characterizing function ties together two mathematical structures possessing analogous operations. One structure is the domain space of the function; the other is the range space. For example, if the measure system being considered is area and we restrict our discussion to polygonal regions in the plane, the domain space is the set of polygonal regions. An operation that provides a structure for the domain space is that of union of nonoverlapping polygonal regions. The range space is the set of positive real numbers. The operation of addition corresponds in some sense to union in the domain space and provides a parallel structure within the range space for the operation of union.

Thus, within a given system of measure, the analysis of the learning must be in terms of (a) the acquisition of the domain and range space structures and (b) the function relating the two structures. Since successful learners use the structure of one space for support and guidance in learning the structure of the other space and in attaining a sense of the function itself, the psychological process under consideration is transfer. Henceforth, this will be called the within transfer problem since it is, so to speak, internal to a given measure system.

Most systems of measure, such as length and area, share one or more characteristics or attributes. That is, the function characterizing a particular measure system, such as length, possesses some properties that are common to the characterizing function of another measure system, such as area. An example of a common property for the measure systems of length and area is that congruent domain elements map to the same range element. Given the extensive experience of children and adolescents with different systems of measure, a remarkable feature of their performance is that it proceeds so inefficiently. That is to say, the typical child does not use the learning of properties for one characterizing function to advantage in learning about other measure systems. Thus, another important aspect of the study of how children cope with measure is an analysis of the learning of one system of measure in terms of the related previously learned measure system. Henceforth, this problem will be labelled the across transfer problem.

A final aspect of the child's dealing with measure stems from the widely held belief that the geometry of measure is one of the best and most appropriate intuitive floors for instruction for concepts of number, algebra, and analysis. For example, multiplication of fractions naming rational numbers is frequently provided an instructional rationale based upon area properties. The addition and subtraction of integers is commonly given a measure-based justification depending upon manipulation of line segments given a directional orientation. Thus, a third structural learning problem deeply involved with measure is also a transfer problem. The transfer is from a measure system context to learning out of the context of the specific system of measure. Henceforth, this transfer problem will be labelled the outside transfer problem.

The outside transfer problem is in many respects a subset of the within transfer problem. It differs from the within transfer problem primarily in terms of the intent of instruction. That is, for the within transfer problem the intent of instruction is to teach about a measure system. For the outside transfer problem, the goal of instruction is not to teach about a measure system but rather the effects of that learning upon other instruction on topics not necessarily bound to a particular measure system. The teacher using measure on the number line to illustrate addition of fractions does not have as a goal to teach length; the objective of instruction is to build skill and understanding for adding fractions. The distinction is in terms of instructional intent and is, admittedly, a somewhat artificial, although useful, distinction.

In this paper the analyses of the problems of learning measure concepts are primarily in terms of the three types of transfer of learning problems, the within a single system of measure problem, the across measure systems problem, and the problem of extending concepts to learning outside of the realm of a measure system. In the next section of this paper, some principles of teaching for transfer are defined to establish a context for examining each of the transfer problems. Then each type of transfer problem is examined in terms of the learning problems associated with that particular category of transfer. Next, some features of measure systems possessing a significant potential for interference with the transfer are considered. As a basis for organization of research concerning the learning and application of measure concepts, a retrospective view of transfer will be given by way of a summary.

Transfer*

The nature of what is to be learned about measure focuses attention on the psychology of transfer of learning. Fortunately, there is a rich tradition of experimental work concerning transfer. This work has its beginnings in the turn-of-the-century attacks on mental discipline and faculty psychology (Kolesnik, 1958) and has culminated in several principles of direct applicability in the classroom. For example, five such principles stated below are based primarily upon the writings of Ausubel (1968), Bugelski (1971), and Cronbach (1965). Even though the statements may lack the qualifiers necessary to assuage the psychological purist's concern for precision, they are stated in terms of instruction in the belief that this specifies the nature of the problems of learning about measure encountered by the child and serves to identify needed research into teaching about measure.

Transfer assumes that two sets of concepts, principles, or generalizations are to be learned. The two sets, P and S, share common characteristics of the subsequent set, S. If the learning of P enhances, improves, or makes more efficient the learning of S, then positive transfer occurs. Transfer is the product of the learning of P. It depends not only on what is learned but how it is learned. That is to say, the conditions for learning P, the stimuli sets and the mediating processes, all contribute to establishing the transfer of learning.

The five principles of teaching for transfer assume that the set of common attributes for the prior learning, P, and the subsequent learning, S, have been identified.

Principle 1. Instructional materials and design for P must be explicit in terms of the attributes of P that are common to P and S.

Principle 2. The more complete and thorough the learning of P, then the greater the probability of transfer.

Principle 3. The design of instruction for S must be in terms of the attributes and conditions of learning that characterize P.

Principle 4. More powerful and inclusive concepts, principles, and generalizations have greater potential for facilitating transfer than the less powerful or inclusive.

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*This section on transfer is an adaptation of a portion of the author's "Mathematical Distinctions in the Teaching of Measure" which appears in the NCTM's Measurement Yearbook edited by Doyal Nelson (in press).

Principle 5. Instruction for S must focus on the differences as well as the commonalities of the attributes of P and S in order to protect the learner from overgeneralization and misapplication of the concepts, principles, and generalizations of P to S.

The emphasis on prior and subsequent learning focuses attention on the temporal factor in consideration of learning. Must children learn about joining two segments on a line (prior learning) before they encounter the additivity property incorporating number into their conception of length? Or does the incorporation of number concepts cue children about properties associated with joining segments? It should be recognized that sometimes transfer happens in reverse order; for example, teachers often encounter a student who, on experience with the intended topic of subsequent learning, finds the prior concept is at last understood. This sort of reverse transfer is not accounted for within the above five principles because this is not the customary intention of planning instruction for transfer. Teaching for transfer is an intentional act.

The temporal intentional factor often is not considered within the instructional design for measure; concepts, principles, and generalizations of the P and S sets are simply muddled together. Textual materials are frequently used that force the child to conceptualize within the domain space and the range space of a measure function simultaneously with the result that understandings are not sharp and useful. Indeed, the time factor may be a variable of some research significance since we do not have an adequate experimental grasp of the effect of this variable. However, the concern of the following sections is with the nature of the content and its implications for transfer.

The Within Transfer Problem

Most measure systems of significance in the school curricula, whether in mathematics, science, or vocational education, are homomorphisms. The application of these measure systems establishes their significance. The homomorphism for a measure system is specified by the characterizing function of the measure system and the structures of the domain and range spaces. For example, if the area of polygonal regions is the measure system under consideration, an area function A maps elements r from the set of all polygonal regions R into the set of positive real numbers $\mathcal{R}+$.

We write:

$$A(r) \in \mathcal{R}+$$

or

$$A: R \rightarrow \mathcal{R}+.$$

Operations can be performed within the domain space; the union of nonoverlapping contiguous regions yields a region. The function A "preserves" the operation of union within the range space $\mathcal{R}+$ as indicated by the diagram:

$$\begin{array}{ccc}
r_1 & \xrightarrow[\text{Function}]{\text{Area}} & A\,(r_1) = 5 \\
r_2 & \xrightarrow[\text{Function}]{\text{Area}} & A\,(r_2) = 7 \\
\hline
r_1 \cup r_2 & \xrightarrow[\text{Function}]{\text{Area}} & A\,(r_1 \cup r_2) = \\
 & & A\,(r_1) + A\,(r_2) = 12
\end{array}$$

The homomorphism provides the basis for the within transfer problem. The characteristics, relations, and operations of interest in the domain space R are preserved in the range space $\mathcal{R}^{+}$ by the function A. The learning of R, its characteristics, operations, and relations, constitutes the prior learning set P for the area measure system. The learning of characteristics, operations, and relations in $\mathcal{R}^{+}$ constitutes the subsequent learning set S.

This transfer situation differs markedly from the traditional description of transfer afforded by psychologists. The traditional definitions of transfer do not have two characteristics possessed by this situation. First, transfer is usually described in loose terms; "P and S share common attributes or characteristics" is typical of word usage relative to transfer. Gagné (1970), for example, in discussing lateral transfer states, "[transfer] refers to a kind of generalizing that spreads over a broad set of situations at roughly the same 'level of complexity'" (p. 335). Note the imprecise nature of the statement. The homomorphism provides a significantly more precise grasp on the nature of the word "share" and, indeed, serves to identify some of the attributes that are common. Second, in a sense the process of transfer itself is one of the desired learning outcomes or teachable objects. The typical transfer task does not have as an objective the process of transfer itself. The characterizing function and its properties for a measure system are, if you will, a set of correspondences mirroring the associations of the transfer process. The learning of a measure system is incomplete unless the child acquires the sense of the homomorphic linking between the domain space and the range space.

Several advantages accrue to examining children's learning processes for measure in terms of homomorphism-based transfer. First, the homomorphic analysis refines the nature of the questions researchers should be asking about the learning of measure concepts. Teachers are not the only ones who muddle together the P and S sets of learning. Second, the examination of homomorphic transfer does provide an argument for the significance of Piagetian researchers' concern for studying carefully the child's prenumerical, manipulative operations with the objects to be measured. Steffe and Carey (1972) and Van Engen (1971) argue, for instance, that many of the measure concepts need to be "operationally defined." To compare the lengths of two rods, the child needs to juxtapose physically the two rods examining the extension of one beyond the other while the other ends are aligned. According to Steffe and Carey, the manipulation of juxtaposition provides an operational definition made by the child for the child. Thus, for many measure systems, operations and relations within the domain set of the characterizing function have operational definitions within the manipulations of objects. The phrase "operational definition" captures both the essence of the manipulative base of the operation in the action of the child and the feel for this aspect of measure being defined by the child in the same spirit of a researcher defining intelligence in the operational sense of performance on an IQ test.

Several different attributes of the domain set may be operationally defined. Among these are:

1. Transitive property. If region A has the same area as region B and region B has the same area as region C, then region A has the same area as region C (similarly, for less area than and greater area than).

2. Substitutive property. If region A has the same area as region B and region B has more (less) area than region C, then region A has more (less) area than region C.

3. Symmetric property. If region A has the same area as region B, then region B has the same area as region A.

4. Asymmetric property. If region A has more (less) area than region B, then region B does not have more (less) area than region A.

5. Reciprocal property. If region A has less area than region B, then region B has more area than region A.

When operationally defined, each of the properties is a prior learning, an element of the P set, for the transfer learning for the measure system of area. The homomorphism "carries" each property into the range space, the set of subsequent learning. The prior learning set for measure, sometimes referred to as primitive subconcepts, accordingly needs careful attention in and of itself. According to Principle 1 for teaching for transfer, instruction for P must be in terms of shared attributes of P and S. The learner who orders regions in the operational sense by using the transitive property has the requisite foundation for coping with order in the more powerful and complete sense mandated by the order properties of the numbers in the range space. This is to say, transfer will not take place if there is no learning set P of operational definitions to provide the base for transfer. Observation of this is frequently difficult since the child has established separately some of the order properties for number. Although the child may appear to possess the range space order properties, they have been established in terms of number and are not part of a measure system tying together attributes of both P and S.

The child cannot be said to understand a measure system until the operational definitions of properties of the domain space, the corresponding properties of the range space, and the characterizing function that unites them into the measure system are exhibited. Many Piagetian researchers err in stating they are studying the child's acquisition of measure when they limit their purview to only the properties of the domain space. This is an incomplete examination of what is involved in learning measure since it involves only the prenumber aspects of measure. This is not to say that the examination of how children cope with the domain space is not important; rather, it is to point out that there is much more to learning a measure system than simply acquiring the operational definitions. Indeed, in this writer's opinion, for length and area systems in particular, we are beginning to have enough evidence of the characteristics of children's learning of operational definitions that we can begin to examine more closely how children tie the range space attributes to the operational definitions. That is, how do children build number into their understanding of measure?

How then should the researcher approach the study of how children incorporate number into their schemata for measure? Analyses of the mathematics and of the psychology of transfer suggest that the unit and unit iteration deserve particular attention. Unit iteration, as the means of construction of the numerical range space, has an apparent logical simplicity when considering length, area, volume, and angularity. The simplicity obtains from the capability of readily expressing unit iteration in terms of corresponding operational definitions in the domain space. The child who conserves regions on joining them or splitting them, in the sense of covering, possesses a base for building an iterative procedure. If region N can be built from regions M and a unit U based upon the child's understanding that N and $(M \cup U)$ cover the same region then a base for iteration exists. Other alternative approaches to building number into the measure system for area appear more complex, either because of the use of numerical properties and operations without a clear operational base in the domain space or dependence upon greater inference of domain space properties from the properties of the range space. Corresponding arguments concerning the efficacy of examining unit iteration can be made for other measure systems such as length, angularity, and volume. The significant questions for researchers point directly to the prenumber domain space and the numerical properties of the range space. Study of how children work within the context of unit iteration appears to be a productive avenue to follow. Gal'perin and Georgiev's study (1969) indicates both the differing conceptions of unit possessed by children and the fruitful character of such research.

Beilin's (1964) study of perceptual and cognitive conflicts concerning area concepts for the young child indirectly addresses some of the learning problems associated with building area concepts on an iterative base. Children were asked to make a judgment of whether two polygonal regions built of units were of equal area. Some of the light board displays of pairs of polygonal regions were of equal area and congruent and some of equal area but not congruent. Children could not manipulate the displays in the sense of using or making an operational definition. The children who were successful in coping with displays of equal but noncongruent displays (see Figure 1) give what Beilin labels as either iterative or translocative explanations. The iterative explanations were apparently

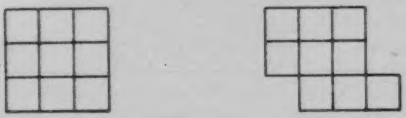

Figure 1. An example of a display from Beilin's study (1964).

labelled as such because of the child's incorporation of number into his explanation. Beilin recognized the problem of the child making positional judgments, concerning the number of units, without any feel for the covering aspects of area. That is, the child's judgments did not preclude using and counting units with little sense that they may be congruent and serve to cover the regions in question. Each unit may have been considered a discrete entity with the nature of its position being more important perceptually than its covering or region-filling nature (Flavell, 1963). The Beilin study is of particular note because it suggests the status of children's thinking relative to the incorporation of number into their conception of the system of area measure. Suggestive of operational approaches that might well be explored further, it identifies some of the quasi-measure factors that the child may use in generating the homomorphic links between regions and numbers. Little research evidence has been gathered concerning the precise nature of the interaction between positional and discrete conceptions of the child and how this evolving interaction contributes to the formation of the mature conception of area for nondiscrete objects. The child's reliance upon the discrete or positional concepts appears to be a factor with regard to whether the child fixates on an inadequate judgment of the area relations between the objects or can indeed be rational in thinking through the comparisons and operations.

The hypothesis that unit iteration provides a superior instructional base for establishing the functional relation between the domain and the range spaces assumes the child has the capability for dealing with operations and relations within both the domain and the range spaces. The transfer analysis also assumes the learning within the range space of numerical relations and operations is new or initial learning. In point of fact, this is seldom the case. Typically the child has had experience with number. Thus, the instructional problem frequently boils down to "How can the teacher design instruction to emphasize and establish the homomorphic character of the measure system?" This may mean incorporating discrete and positional concepts into the instructional strategy for teaching area with the intent of withdrawing or extinguishing the child's reliance upon them as concepts of reversibility and conservation of the measure system are acquired and linked with number concepts.

The instructional sequence for area in many text series shifts from the unit iteration approach to the rectilinear product approach specified in terms of length times width. This may well happen to some children before they sort out and clarify many of their early conceptions of the unit and make the homomorphic link between regions and numbers.

The hunch of the author is that many of the primitive subconcepts or properties of the domain space are obscured for the child by the fact that every polygonal region must be forced into a rectilinear representation. The child must use the primitive subconcepts in an operational sense to acquire the concepts and skills that are the targets of instruction. To cut up a parallelogram into pieces and rearrange them to form a rectangle demands conservation. The rearrangement forces the learner to make decisions concerning the best way to partition the figure. Indeed these decisions demand more than an understanding of conservation. There are many ways to partition a parallelogram. Some of them "work" to produce a nice rectangle; some do not. We need more research of the nature of Wagman (1975) that examines the relation of conservation to the partitioning problem. Moreover, the confounding factors of decision-making of the nature discussed in Wertheimer's Productive Thinking (1959) should not be ignored. The difficulties of some mature students in constructing proofs of the parallelogram area formula within a theorem sequence that demands consideration of the cases in which the altitude falls outside, as well as inside, the base is indicative of the conceptual decision difficulties extending beyond conservation.

In using the rectangular product approach to area, a second type of difficulty that obscures the primitive subconcepts for many learners is that the correspondence between the union property in the domain space and the addition in the range space must necessarily incorporate usage of the distributive property. Weaver's research (1973) suggests many children may not naturally have facility with the distributive property to a sufficient extent for it to serve as an instructional base. Careful task analysis reveals that a logical organization for instruction would require that the child already possess some sense of the rectangular product definition of area before being able to grasp all that is involved with the additivity property when confounded with distributivity.

The learner needs to acquire control of the rectangular product approach to area. It is a practical and expedient understanding to possess. The unit iteration approach to area does not have sufficient power or efficiency to allow the student to cope with area in settings involving nonpolygonal regions or problems involving irrational numbers. A learner's understanding of area typically evolves from an intuitive informal feel for units, covering, and iteration to a more thorough and complete basis. The move to the product based area in terms of rectangular regions frequently amounts to an abrupt shifting of the student from familiar iterative settings to settings demanding new intuitions.

Text series tend to ignore or evade the problems of helping learners establish the connections between the iterative based and the product based approaches to area. The suspicion of the author is that many of the resultant problems of learners in coping with area derive from the nature of transfer; that is, teaching for transfer requires attention to the how of learning as well as the commonalities of what is to be learned. The referents for the iterative instructional approach appear different to kids than those for the product approach. (See the Carpenter paper in this collection.)

The learner typically does not develop his intuitions concerning the rectangular product version of area measure in an official measure setting. Rather, his primitive notions develop within the curricular setting of number concepts and skills. The product-area approach is used an an embodiment, or a referent base, for teaching concepts and skills for whole and fractional rational numbers (the outside transfer setting). Often there is no diagnostic check to ascertain whether the child does possess sufficient understanding of area for this instructional appeal to intuition. For many children, this incidental setting is where area measure concepts are acquired. The perceptual-conceptual basis for the multiplication of numbers typically is of a discrete, positional nature. Teachers and designers of the instructional materials for multiplication typically do not attend to area as the rationale or embodiment for multiplication. The child's chubby finger moving from square to square in a 2 x 3 rectangular lattice is sufficient for the child to register, on a positional and discrete basis, that two onto three is six. But it does not necessarily help the child associate each of the squares with the space-filling character of these six units. Consequently, the embodiment of product concepts on an area base often lets the operationally defined domain attributes for regions slip away if they have been established at all.

One of the outcomes of this incidental learning about area develops later when the child officially encounters the area formulae. An assumption of prior experience with area is made; children are simply given the formulae. Such a development appears to leave to chance whether the child will relate the domain space attributes or the operational definitions for area to the computational formulae developed within the rectilinear product curricular strand. The oft-stated remark that teachers tend to force the computational measure formulae on students too early may well be true in many cases. In fact, it often may not be a matter of formulae being taught too early, but rather of the formulae being established relatively independent of the conceptual bases of the domain space and its homomorphic link with the range space.

Students eventually must use and understand the product approach to area. Study of how the child builds the product concept of area into his thinking is important. The roles of the operational definitions, the domain space manipulations, need to be examined both in terms of their specific contributions to the child's acquiring the more mature product based area concept and in terms of whether the product area concept forces changes in them. But, the within transfer analysis is complicated by the need to shift from the unit iterative approach to the product approach for incorporating number into the area measure system.

The within transfer problem has, to this point, been discussed in reference to the child's conception of area. This measure system was selected primarily for two reasons. First, area involves an intriguing and revealing interaction of the psychological and mathematical factors in considering the learning of children. Second, area concepts provide the intuitive geometrical gateway for a large number of other important mathematical ideas.

Many other measure contexts could serve as the domain of discourse for examination of the within transfer problem. The particular contribution of the Piagetian psychologists' analyses of the child's conception of measure has provided extensive evidence supportive of application of the first principle of transfer, stated previously. They have refined the notions of the operational factors in the child's thinking within the domain space of the characterizing function. The many studies of distance and length provide a firm foundation for establishing the homomorphic character of these functions in terms of designing instruction for the prior learning set or the domain set of the function. The distance function operating on a domain set of segments such that each segment is mapped to a positive real number has been explored thoroughly by researchers in terms of the child's operationalizing definitions based upon manipulation of segments and objects. Strict analogues of the attribute statements for area discussed earlier have been explored in great detail. In addition to these properties of the domain space of the function, the following three properties of the function are important:

1. Additivity: The join of two nonoverlapping segments has the same measure as the sum of the measures of the individual segments.

2. Unit: There exists a segment that maps to one in the range space.

3. Congruence: Congruent segments map to the same real number.

The same primary problem exists for researchers that was described for the area measure system. Namely, "How does the child incorporate number into his schemata for distance?" Iterative procedures again appear to have a nice potential for establishing the character of the range space and its relation to the domain space. Again, the discrete and positional factors in children's reasoning and perception appear to confuse their incorporation of number into learning about distance. The child who concentrates more on the number of units and considers them as discrete positioned "points" in working on the number line without using their covering attribute does not possess a measure system of notable power. Must the child go through this stage in his reasoning? Is it necessary for him to do so in order to incorporate number into his schemata for distance? Does it ever lead to proactive inhibition in his tying together range and domain space attributes?

Although no precise parallel exists to the problem of how children incorporate the product concept of area into their iterative base, the problem of how children move from "nice" polygonal regions to the more complex curves is paralleled. In the upper elementary and junior high school years the child must acquire a sense of distance on a curve. Little evidence of how children learn to link number with curves that are not nice straight lines has been collected.

The volume function also yields to a homomorphic based analysis of transfer. The properties of unit, congruence, and additivity hold as they do for distance and area (in the across transfer sense). In the within transfer setting, the critical questions concern how the child establishes the homomorphic "links" between the domain space and the number or range space structures. If prior research of a Piagetian nature is examined, our knowledge of how children conceptualize the operational definitions or the prenumber base for comparison is more than a little confused. For example, Piaget, Inhelder, and Szeminska (1960) defined stages corresponding to the child's conception of volume using the descriptive phrases interior volume, occupied volume, displacement volume, and mathematical multiplication. The last stage corresponds to the incorporation of number into the volume homomorphism model in the same sense as the product approach to area. Displacement volume conjures up images of Archimedes' Eureka and specific gravity. (Some researchers appear more than a little confused by the distinction between specific gravity and displacement volume, failing to recognize when children make more accurate scientific observations than the researchers do in their analysis of the children's statements. See Kamii and Derman (1971)). Interior volume is defined as the invariance of the amount of matter which is contained within the boundary surfaces, whereas occupied volume is defined as the amount of space occupied by the object as a whole in relation to other objects round about (Piaget, et al., 1960, pp. 359-360).

Even though the mathematical distinction between interior volume and occupied volume seems hazy, Lunzer (1960) and Elkind (1961), like the Geneva group, have observed stages in children's reasoning corresponding to these stages. The differences in the actions and statements of children appear to hinge on how they incorporate number into their thinking about volume. Unlike research in area and distance concepts, the volume situations that Piagetians have posed for children do not readily allow the child to operate exclusively in the domain space without resorting to number either in the iterative or multiplicative sense. Thus, the status of the child's operational definitions is difficult to determine or observe. Engleman's (1971) approach to displacement seems more clearly delineated in this sense although volume was not his primary concern. Although the parallels between the area measure learning and the volume measure learning are striking in terms of the within transfer analysis, the evidence concerning the child's operationalizing concepts in the domain space or prior learning set for volume are not available.

The disparaging tone of the previous paragraphs concerning the lack of clarity of definitions concerning the four stages of the child's conception of volume may be symptomatic of an unavoidable complexity; perceptually, volume is a mess for the child. The child can operationally define, through juxtaposition, his conception of comparison for both length and area. But how does a child juxtapose two volumes? Whether they be liquid or solid volumes, the child encounters manipulative and/or perceptual complications. Pikas (1966) and Wohlwill (1962) identify spatial and temporal continuity as key variables in concept formation, particularly for the young child. The perceptual and manipulative difficulties suggest that the child's use of numerical cues in forming the concepts of volume may be productive for researchers.

Designing activities that focus directly on the operational definitions for solid and liquid volumes for the purpose of assessing children's thinking or for the purpose of designing experiments appears to be unrealistic because of perceptual complexity. Unless an iterative or multiplicative situation that incorporates number exists, one cannot compare volumes without resorting to the indirect measure context of displacement. The use, by some children, of numerical cues in the volume measure context prior to their having a complete multiplicative grasp of volume is established, but the precise nature of how the child acquires and uses this range space information is not known (Carpenter, 1975). Lovell and Ogilvie (1961) conjectured that learning the volume formula for rectangular parallelepipeds would enhance the acquisition of the conservation of volume. In order to gain more complete information about how children build the homomorphic linking of the domain space and the range space for the volume function, we need to examine more carefully if and how children use numerical cues from the range space to build conceptions in the domain space.

Studies of the nature of Carpenter's (1975) suggest that experience and maturity are the key factors controlling children's acquistion of volume measure concepts. Cross-cultural Piagetian studies, such as Gay and Cole's The New Mathematics and an Old Culture (1967), also suggest the key role of experience factors in the child's conception of volume although their work does not directly examine the child's operational definitions in the sense stipulated in this paper. Their study of children in the Kepple culture in Liberia focused upon the children's experience with conically shaped piles of rice in the market place or in the preparation of food. Their observations indicated simply that Kepple children were shrewder in judging volumes than their age-group counterparts in the United States.

These studies were not, however, designed to reveal the effects of incorporating numerical cues into the child's experience. The fact that maturity and experience are apparently the key factors, if learning is judged in terms of domain space concepts alone, suggests turning to other variables. Thus, the study of the young child's coping with volume measure appears profitable for curricular development in the context of their use of numerical cues to form and refine their operational definitions. The complexities of the perceptual context and the necessity of the child's using indirect comparisons suggest that children's acquisition of volume concepts is most profitably examined if a larger base of experience with measure exists in the within transfer setting.

Volume measure has been contrasted with distance and area measure in terms of the possibility for direct comparison within the domain space for the characterizing functions. Clearly length and area provide settings for the child to use direct perceptual evidence to build operational definitions. This is also true for measure systems for numerousness (discrete objects) and for angularity. The large majority of other measure systems (e.g., speed, temperature, et cetera) do not allow for the child's building operational definitions upon the simple direct perceptual evidence of their manipulation of physical objects. The within transfer analysis is most applicable to those measure systems characterized by the effects of direct perceptual evidence in the child's conception of measure.

The Across Transfer Problem

The across transfer analysis of the learning of measure systems assumes more than one system of measure is to be learned. Piagetian developmental psychologists have examined this type of learning but from a relatively narrow vantage point. The questions have been oriented toward investigating whether concepts, such as conservation, reversibility, and transitivity, occur at the same time for two or more measure systems, or whether there is a natural order depending upon the type of measure systems being studied. This type of research is usually labelled with the descriptor "horizontal décalage." Examination of studies within this realm leads to the conclusion that acquisition of these concepts is specific to a measure system; for example, if a child conserves in one measure system setting, one cannot assume that the child will conserve in a different measure system. Second, there appears to be a weak natural ordering of acquisition dependent upon the type of measure systems. But very few studies have been conducted that look specifically for the effects of learning in one measure system on the learning of another. Parenthetically, it should be remarked that this would be most useful for curriculum designers and the writers of textbooks and instructional materials.

One of the few studies that examines the effect of learning in one measure system on learning in another measure system is Montgomery's study (1972) of aptitude treatment interaction. Interested in children's acquisition of area concepts, Montgomery specified aptitude in terms of the children's ability to learn concepts related to units of length. That is to say, the design of the experiment analyzed learner performance in one measure system in terms of performance within another previously learned measure setting. Second- and third-grade children were given a pretest, a brief period of instruction concerning length, and finally, a posttest concerning length. Based upon performance, the children were split into two levels. Subjects in each level were randomly assigned to two different area instructional treatments, one of which emphasized unit and iteration while the second did not. For achievement and retention measures, she reported significant effects due to treatment (favoring the unit approach) and aptitude but no significant interaction effects. One of the two levels of aptitude consisted of children who learned about length in the pretreatment common unit, the other consisted of children who exhibited but a small amount of change in performance on the length tasks. A third aptitude group of children who had the length concepts both before and after the pretreatment unit had been hoped for, but simply did not exist.

One interpretation of this experiment is that it is a study of across measure systems transfer. The criterion variables may be considered from the point of view that they are symptomatic of ease or efficiency of learning about area (the S set of learnings) in terms of the prior (or P) set of learning about length. The particular attributes of the P and S sets that were focused upon concerned units and their characteristics. This is not the standard interpretation of transfer although it is quite similar to the spirit indicated in Cronbach's Issues Current in Educational Psychology (1965). In terms of across systems transfer, one treatment focused upon the common attribute of unit. The nonunit treatment as described appears to be a soundly constructed instructional program that does, indeed, stress the unit idea by repeated usage of congruent units. But this treatment does not explore alternatives of using different units to measure the same region and the comparison of the resulting numbers. The unit approach focuses upon characteristics of the unit and results of modification of the unit.

The ANOVA revealed differences in achievement significant at the 0.0001 level. Comparative performance, in terms of raw scores, is indicated by Figures 2 with the upper curve representing the unit treatment. The I and II indicate levels of aptitude in coping with linear measure.

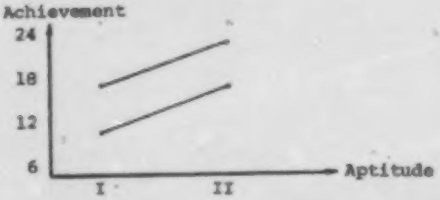

Figure 2. Aptitude treatment interaction graph from Montgomery (1972, p. 94)

The retention tests results, given approximately a month later, paralleled those of the achievement test. A transfer test including items on numerousness and volume measures was given, but due to poor test reliability, did not provide interpretable results. (Note that this is the more traditional treatment of transfer. Perhaps a better question is to examine transfer to volume in terms of ease and efficiency of learning in an instructional context.) An interesting side result was that children in the unit treatment were more inclined to recording the unit used in assigning the measure than children in the nonunit approach, even though the matter of recording had received the same emphasis in both treatments.

Montgomery indicates (in a letter, 1974) that she has used the length pretests and the unit area treatment in an interesting fashion. Half of the children in three third-grade classes were selected randomly and given the length pretest. Then two of the classes were taught the unit area treatment and the third class was given an arithmetic unit. All children were then given the second length test that had been used in determining aptitude. Eyeball examination of the raw data indicates the area instructional treatment helps kids in learning about length. The difference in mean performance for the area treatment groups on the length test was 4.14 and for the arithmetic treatment group 2.2 with the data suggesting that the taking of the pretest contributed to the magnitude differences in the scores on the twenty item tests. This evidence is weakly supportive of attention to transfer in terms of specific concentration on the attributes shared by two measure systems.

The experiments reported above provide evidence of a potential in examining the learning in one measure system in terms of its effect on the learning of instruction in a second. Neither experiment was designed in terms of transfer although the former provides a relatively good base for interpretation in terms of ease and efficiency of learning and, consequently, instruction. The interjection of two instructional units concerned with the second measure system provides a means of acquiring insights into the nature and effect of the particular attributes common to the prior learning set and the subsequent learning set. Clearly, instruction in length, the prior learning set, was designed to emphasize the attribute of unit common to both learning sets (Transfer Principle 1). In the same sense, for the unit treatment, the subsequent learning tasks were oriented in terms of the common attribute of unit (Transfer Principle 3). The main effect differences reported for aptitude can be considered as an exemplification of the second transfer principle. Utilizing two instructional treatments for the second measure system establishes a setting for identifying operant factors in the second measure system.

The Developing Mathematics Processes (DMP) materials designed by the Wisconsin Research and Development Center (Romberg & Harvey, 1974), seem particularly well designed for experiments concerning effects of different strategies concerned with units in measure systems. Attributes of objects that are readily isolated perceptually are examined and measured with concerted attention given to units.

The units considered for a given system of measure are of two types: (a) arbitrary and nonstandard and (b) the standard units. The measure systems encountered are length, numerousness, time, weight, area, and capacity. Thus, a convenient context for some productive research is provided within the setting of a carefully considered sequence and scope of measure related objectives and activities. One would hope for studies, comparable to Montgomery's in design, that would be concentrated on exploring the transfer effect from learning about units and their properties in the length and area settings to learning in the settings of liquid and solid volume, time, and weight.

Mathematical analyses of different measure systems reveal several attributes that are likely candidates for this type of across measure systems transfer research. Blakers' Mathematical Concepts in Elementary Measurement (1967) provides extremely detailed mathematical analyses of this sort. Given a characterizing measure function, m; a domain space, D; and a range space of real numbers, *R;* the following loosely described attributes are significant mathematically and are shared by many measure systems:

1. Unit and iteration. A special $u \in D$ such that $m(u) = 1$ can be defined for most significant measure systems. It allows the "covering" of any element $d \in D$ in the sense paralleled by the Archimedean property of the range space.

2. Additivity. $m\ (d_1 \cup d_2) = m\ (d_1) + m\ (d_2)$ for d_1 and d_2 nonoverlapping.

3. Congruence. $d_1 \cong d_2 \quad \rightarrow \quad m(d_1) = m(d_2)$.

4. Comparison. $d_1 \subset d_2 \quad \rightarrow \quad m(d_1) < m(d_2)$.

These four basic properties are each likely candidates for the design of research for the across systems transfer problem. It should, of course, be noted that they are interrelated in each measure system. The Montgomery study relating to units would have yielded nonsensical results if the children had not possessed a sense of the congruence property for the given units under consideration. But, clearly, transfer studies can be designed that give special emphasis to each of the identified attributes. It should also be remarked that the comparison attribute contains as related subattributes the domain properties described as subject to operational definitions by children in the within transfer problem described earlier.

Each of the transfer principles offers the potential as serving as an identifier of hypotheses for research. For example, the principle stating that the differences between prior and subsequent learnings need to be emphasized suggests the following across system transfer problems. The measure systems for length and temperature appear to share several characteristics, but one that is not shared is additivity. If $80^{\circ}C$

coffee is added to 25°C coffee, it does not start to boil. What is the effect on the learning of a temperature measure system by concentrating on this critical difference? The length and area measure systems each possess the congruence attribute, but for area the converse statement is false and for length it is true. What impact on the learning of one system does emphasis of this difference have, given either order of encounter with the systems?

A final note is in order about the across systems transfer problem. Each of the measure systems considered has been simple. That is, length, volume, area, angularity, and temperature each involves but a single function. Many of the measure systems that provide understanding and control of the environment for both the child and the scientist are complex; in these systems the domain space is a set of ordered n-tuples each element of which derives from a simple measure system. For many of these functions, each of the entries in the n-tuple is from a different measure system (e.g., rate is a function of distance and time). Clearly, it is possible to design experiments similar in structure to the previously described across transfer problem experiments. For example, the structure of the work measure system (foot-pounds) is exactly parallel to the structure of a heat measure system (British thermal units). Because of the identical structures of the measure systems, these two systems could serve as the base for such a research design. However, designing transfer experiments on such directly analogous measure systems probably is not as potentially helpful to curriculum designers as the examination of transfer problems across systems that are not so directly parallel in structure.

A psychological problem for learners that is quite troublesome is described with the word centration. The word refers to the propensity of the learner to center attention on the perceptually dominant feature of an object ignoring other salient features. For example, in the comparison of occupied volumes, the child may determine his comparative conclusions by attending only to the heights of the objects under consideration. In a comparable fashion, when dealing with the measure system for rate, the learner may consider only the effects of modification of time without considering the effect of a change in the distance characteristics of the system.

This suggests a model for experimentation in the across systems model that is a bit different but which may be revealing. Consider two measure functions, M_1 and M_2, each of which has a domain that is an n-tuple. For M_1, all of the domain elements are from elements of the range of a single measure system. That is:

$$M_1: (X_1, X_2, \ldots, X_n) \rightarrow R \text{ where } x_i = f(z_i) \text{ for } f \text{ a measure function.}$$

For M_2, not all of the domain elements are elements from the range space of a given measure system but are drawn from different measure systems. That is:

$$M_2 : (Y_1; Y_2, \ldots, Y_n) \rightarrow R \quad \text{where } y_i = f_i(x_i) \text{ and } f_i \neq f_j \text{ for some } i \text{ and } j.$$

For example, learner's performance in coping with distance as the product of rate and time might be compared to their performance in handling area as a product of length and length. Across transfer studies designed on this comparative functional structure may reveal interesting aspects of children's propensity to centrate. Experiments focusing on centration in newly encountered measure systems, such as rate, might produce interesting and productive results if the learner has a base of the somewhat more simple situation of the arguments all coming from the same domain space. In point of fact, we have little or no evidence concerning how children learn or conceptualize what we frequently called derived systems of measure in which the domain is an n-tuple drawn from several different measure systems. For instance, the Piagetian observations of how children handle speed and rate reveal that children find difficulty in these situations but do not indicate the nature of the difficulties. By using the area and volume functions, with children having control of the multiplicative character of the function, a base for designing potentially revealing experiments exist.

The across-transfer problem is suggestive of many research studies. Few studies have been conducted in this area. The structures of measure systems are predictive of the nature of, or models for, such studies. They do have the advantage of utilizing the findings of the Piagetian-style developmental research that identifies those operational definitions possessed by children for domain spaces for each characterizing function.

The Outside Transfer Problem

It is frequently stated that measure is ubiquitous, pervading all we do in mathematics and science. Within mathematics this is reflected by the fact that measure concepts are used so frequently as the intuitive base for instruction for nonmeasure concepts and skills. For example, measure ideas are popularly selected as the embodiments for instruction about operations with numbers, characteristics of function, limit concepts, and the like. But the purpose of using these measure based embodiments is not to teach about measure; rather, the instruction is directed toward nonmeasure learning. The purpose of this section is to consider some of the lacunae in our understanding of how children use measure concepts and understandings in this outside setting.

The outside transfer problem is closely related to the within transfer problem. It differs in that for the within transfer problem the goal of instruction is measure, that homomorphic linking of conceptions of the domain and range space structures. For the outside transfer problem, the instructional goals are the skills and understandings for which measure serves as the embodiment. In the case of operations with fractions, the measure embodiments provide the examples used by teachers for an intuitive entry to the concepts and skills for the numbers. It should be recognized that the domain of discourse shares many characteristics with the proverbial chicken and egg problem. Which come first, measure concepts or number concepts? The character of the child's first encounter with numerousness is in the sense of measure applied to discrete sets of objects. This has been explored by researchers in a thorough fashion. The concern of this section is, in particular, to examine measure related learning problems for number systems in which teachers will want to use more sophisticated measure systems than numerousity.

First, consider one of the oft-stated remarks about conversion to the metric system. A purported advantage of conversion is that learners will use the decimal system of measure and that emphasis on computation with vulgar fractions may be cut back in the elementary and middle school curricula. This is a misleading statement; it obscures some of the real questions concerning how children acquire computational proficiency as a result of operating within a measure setting. The significant questions do not relate to the fact of the metric system's computational base of decimal fractions. Rather the question should be, "How can the measure concepts and understandings unique to the metric system be used to advangage in helping children become more accurate and efficient in handling decimal fractions?"

Bauer (1974) conducted a study of three different instructional strategies relating to decimal computation. Her population was 568 seventh-grade students in the 18 classes of six teachers. The three instructional strategies differed primarily in terms of the definitional base for decimal fraction and the appeals to that base for the rationale for the operations. One definitional base was stated and exemplified in terms of measure on the number line. Another base was the more standard appeal to the vulgar fractions with powers of ten in the denominator. The third definitional base utilized place value concepts. The statistical analysis of the data revealed no significant differences in performance. But it is interesting to observe that the subjects had all just completed an extensive unit on common fractions and related computation. Why was there no superiority derivative from the immediacy of this experience with fractions? The definitional base utilizing measure concepts on the number line utilized many of the attributes of the distance function for rationalizing addition and subtraction. The multiplication of decimals was provided a rationale based on the characterizing area function much in the sense of Green's diagram approach (1969). Although not conceptualized in terms of transfer, the Bauer study indicates the strong potential for designing instruction on this base. One must wonder what the results would have been if instruction had been designed specifically for transfer both when the children were encountering the basal measure concepts and when they were dealing with the decimal fractions. Although the experiment did honor the metric system, it was in the sense of a transfer to metric test subsequent to the instructional unit. Would it have been wiser to have developed basic length and area measure properties within a sequence involving the metric system in part and then looked for transfer effects to decimal computation? Or what would have been the outcome of incorporating metric system concepts and skills throughout the computational instructional treatment?

Generally, research into the instructional strategies for computation for numbers other than whole numbers has not attended sufficiently to the intuitive conceptual bases that provide the rationale for the operations. In particular, if the rationale is based on measure concepts and understandings, insufficient attention has been given to the learning context for the measure concepts and understanding and to the common attributes of the two learning sets in terms of how the transfer to number occurs. This is to say, the first principle of transfer is ignored or relegated to a position of unimportance in the design of the instructional strategy for computation. Indeed, an exhaustive review of the literature reveals little research that analyzes the learning of the basal measurement concepts on which the instruction is grounded. There is a body of research that attends to the intuitive measure base and its effect (for example, Green, 1969), but its focus is not in terms of the nature of the measure base possessed by the learners.

The research chosen for the primary example for this section concerned computation with decimal fractions. It would have been just as appropriate to have selected examples of research concerned with vulgar fractions or real numbers. The careful identification of attributes of the learning of measure concepts and learning within the domain of number suggests many transfer designed studies. Aptitude treatment interaction studies have not revealed very much, in this writer's opinion, for they overlook the processes of the prior learning to focus only on the product of that prior learning. The payoff in terms of researching the transfer effects from measure learning to understanding of number may well be more a function of how the learning takes place than simply what the child knows. It should be noted that many researchers, but not all, in mathematics education appear presently to have an aversion to examining the problems of learning concerned with numbers more complex than the whole numbers. This is unfortunate. The problems of how children relate measure to number is fascinating, and we have yet to develop a research base for the design of related curricular materials.

Factors Confounding Measure Learning

The purpose of this section is to list some factors that confound the learning of measure. Some of these relate directly to the transfer process and have the potential of interference with designing instruction for transfer.

The word measure has been used in a very restricted sense in this paper. Except for brief references, such real world measure ideas as conjured up by words such as metric, English measurement system, ergs, joules, hectares, coloumbs, and the like have been ignored. This paper has discussed measure in the context of the idealized mathematical model of real world measurement. In this section, we will specifically limit use of the word measure to the ideal world of the mathematical model and the word measurement will signify the nice but messy world of inexactitude in the real world measuring of the scientist or the proverbial man-on-the-street. Thus, the word centimeter refers to a measurement system; the word length to a measure system.

The above distinction is important. Measurement is an inexact observational process concerning attributes of perceptual reality of the child. But the inexact nature of the observational process and its refinements have the possibility of confusing the basic idealized measure concepts that serve the learner as intuitive foundation for learning about number, for developing geometrical concepts, and for acquiring many other important mathematical ideas. To confuse the child's operationalizing the additivity concept for the length function by noting that a 5 inch and 3 inch measurement each has a confidence interval of 1/2 inch and that errors are additive is regrettable. But such concepts as relative error, precision, and accuracy are important. The potential for mutual interference of measure and measurement concepts has not been well worked out, and the impact on learners has not been established. Presently, the inclination of many textbook authors appears to be to delay the introduction of a careful treatment of measurement concepts until the learner is more mature. But the problem exists. For example, children use rulers that only yield rational numbers, even though in the idealized model there is a single real number corresponding to a pair of points that may be irrational. This necessarily is somewhat confusing and ambiguous. Clearly, the ambiguity has the potential of confounding transfer effects to the detriment of children's learning. It is particularly evident in the design of some devices designed for teaching measure concepts. One commercially available device designed for teaching the set of measure "facts" concerning the angles formed by chords, secants, and tangents cutting a circle is designed so that the child who reads the scales carefully is highly unlikely to arrive at the desired conclusions. The learner may be off by as much 10° in summing his measurements. This is to say that beyond a certain point, it is critical to establish appropriate understandings and distinctions between measurement (as an observational process) and measure (as an ideal concept in a mathematical model). But little evidence exists concerning the timing and the means of establishing these distinctions. The distinction is sophisticated, indeed, bearing upon one's philosophies of science and mathematics. But the child must use measurement to generate his perceptual base for forming concepts.

Another confusing factor in the child's learning of measure is the language of measure. For most measure functions, the word used to identify the function is also used when talking of the range elements of the function. For example, we talk of the distance function and the distance between two points. Distance is a number; it is also the function. But the problem really comes home to roost with angle measure. For most measure systems we can distingish by word choice between the range space measure and the attribute of the object being measured; segment, length, and distance, have a nice characteristic clarity in our speech. But how many teachers refer to an angle of 45°, meaning either the measure or the angle. The best we can do is talk in terms of angularity or carefully use angle as a modifier for measure if referring to the range space elements. These linguistic distinctions have a potential for confusing children.

The perceptual "stuff" of measurement on which the child bases his conception of measure is simple for measure systems like distance, area, angularity, and, to some extent for volume. (Previously, the complexities of the occupied volume, capacity, displacement volume, multiplicative volume, liquid volume, and solid volume were addressed.) But, for other measure systems, the attributes are measured with tools that provide at best indirect evidence of the attributes being considered. For example, in coping with mass or weight, the child either (a) reads a scale or (b) observes whether a balance beam is balanced. In the latter case, not only is the perceptual evidence removed physically from the object being weighed, but the child must incorporate evidence of what he does to the other pan into his perceptual schemata. Consider the case of temperature; the attribute of the length of the mercury column and the position of its endpoint on a number scale bear little (no) resemblance to the attribute of the object under consideration. Thus, we have two different perceptually loaded situations: (a) The objects of concern are removed or at a distance from the perceptual evidence used by the learner, and (b) the perceptual evidence used by the learner may have little "resemblance" to any obvious attribute of the object. The learning of such measure systems having one or more of these characteristics may well be enhanced by an across transfer approach to instruction. But the confusion arises when (a) the perceptual base for the within transfer approach is confusing the learners and (b) the use of measurement tools necessarily involves the learner (and the teacher) with problems of relative error and accuracy.

Retrospect

The focus of this paper has been transfer of learning. Analysis of the mathematics of measure systems indicates the structures of measure systems share many commonalities. Of great significance is the homomorphic character of many measure systems.

The homomorphism of a single measure system defines, in a sense, the within transfer problem. Previous research into how learners acquire measure concepts has not given sufficient attention to exploration of how children form the homomorphic linkage between the domain space and the range space. Rather, the concentration has been on children's conception of the domain space. For perceptually complex measure systems, such as volume or mass, the limiting of the study of children's thinking to the domain space is not productive. Shifting the research model to a homomorphic analysis of transfer in order to examine the incorporation of number into measure appears to be a productive direction to follow. Any examination of the learning of measure is incomplete if it does not explore the child's building the homomorphic links between the domain space and the range space.

If you accept the task of examining the research literature concerning the learning of measure concepts, you realize little is known except for the preliminary encounters of children with the more perceptually simple measure systems of numerousness, length, and area. Much of this research has been in terms of domain characteristics. The more complex measure systems, such as volume, mass, rate, energy, and work, have not been explored.

The more complex systems typically do not yield perceptual evidence of direct use to the child in building operational understanding of measure. There is a profound need to explore how children use the common structural ideas across measure systems to build understanding in the perceptually complex systems. The base for designing instruction for transfer exists, but we have not explored the payoffs and pitfalls of teaching for transfer across measure systems.

The transfer approach has one distinctive advantage in considering the learning of measure. The tradition of the Piagetian developmental research is passive. The status of children's conceptions is observed. This is useful information. But it does not help the teacher decide what should be done to help a child who is having difficulty or yield much prescriptive information for the design of the curriculum. However, the transfer orientation necessarily incorporates an interventionist point of view This is potentially useful from both the practical standpoint of improving the curriculum in the schools and finding out what operates in children's conceptualization about measure. A scientist must intervene to tilt a system if critical variables are to be identified or observed. Static observation alone may overlook a significant variable. In any equilibrium system, forced perturbation of one characteristic may cause shifts in an undetected but critical characteristic. A noninterventionist's observational philosophy removes many possibilities for observing perturbations.

Finally, much of the learning about measure is of an incidental nature in today's schools. Measure concepts are encountered in settings where the goal is teaching and learning about number. It is assumed that measure is intuitive and sufficiently possessed and understood by the learner to serve as an intuitive embodiment for explaining numerical operations. This assumption should be questioned. Further, the nature of how children learn and use measure in this outside transfer setting needs careful attention.

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Analysis and Synthesis of Existing Research on Measurement

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A fundamental question in the learning of measurement concepts is whether the conceptual development of measurement in young children parallels the logical development of the measurement process. The mathematical construction of a measurement operation consists of two basic steps: premeasurement, establishing empirical procedures for directly comparing, ordering, and combining elements of some domain of elements that possess a given attribute; and measuring, defining a measurement function which assigns a nonnegative real number to every element in the domain in such a way as to preserve the essential characteristics of the domain (Blakers, 1967).

Some of the first research to demonstrate that premeasurement concepts are a significant factor in the development of measurement was reported by Piaget and his associates (Piaget & Inhelder, 1941; Piaget, Inhelder, & Szeminska, 1960). Their description of the development of measurement concepts has provided the focus for most of the recent research on measurement learning. Of greatest interest has been their description of the development of conservation and transitivity. As a consequence, most recent studies have dealt with these premeasurement concepts. Fewer Piagetian studies have considered measuring per se, but the studies that have done so provide some fresh insights about learning measurement concepts.

While Piaget has provided a major unifying influence for measurement research, there is also a diversified collection of studies that is not based on any well-defined psychology theory. This collection includes (a) a variety of status studies testing children's skill in measuring or their familiarity with basic measurement terminology and (b) a number of studies investigating the effectiveness of various instructional strategies. Because these studies lack any unifying conceptual basis, it is difficult to assess their total contribution, and they are only surveyed briefly. The primary focus of this paper is the measurement research based on the studies of Piaget and his associates.

Piaget's Measurement Studies

In the discussion of the measurement process, Piaget identifies a number of significant issues. Foremost is the concept of conservation. For Piaget et al. (1960) the central idea "underlying all measurement is the notion that an object remains constant in size throughout any change in position" (p. 90). Thus, the attainment of conservation and the corresponding notion of transitivity is the hallmark of the first level of achievement of measurement concepts. The development of a metric is further dependent on the synthesis of subdivision and change of position.

> Measurement begins when one part A belonging to a whole C is compared with the remaining parts of the same whole by change of position (either its own or that of a common measure, used transitively) so that A (or its equivalent) is superposed on these other parts. This implies that subdivision and change of position are fused into one single operation and no longer simply complementary. That operation is unit iteration. (Piaget et al., 1960, p. 399)

Finally, the development of formal measurement operations is completed with the onset of the ability to calculate areas and volumes from the respective linear measures, which is dependent on the development of the ability to coordinate the measurement of several linear dimensions. Thus, according to Piaget the conceptual development of measurement concepts parallels the logical development of measurement operations in that premeasurement operations are a prerequisite for the subsequent development of a measurement function.

Stages of Development

Piaget et al. (1960) divide the development of measurement into four stages, with the second and third stages each being further divided into two substages. The earliest stages (I and IIA, up to ages of 5 or 6 years) are characterized by a "wide variety of responses which have only negative characteristics in common" (Piaget et al., 1960, p. 117). Measurement is not possible in these stages because space is not viewed as a common medium containing objects with well-defined spatial relations between the objects. Due to this uncoordinated view of Euclidean space, a preoperational child does not recognize that the distance from A to B must equal the distance from B to A. Children also believe that the distance between two points decreases when an object is placed between them, because some of the space has been taken up by the solid object.

In Stages I and IIA, children do not conserve length, area, weight, or volume; they rely strictly on one dimensional perceptual judgements. Length, for example, is judged strictly on the basis of end points. Undulations and angles in the objects being compared are generally ignored, and segments with unaligned endpoints are judged to be unequal. Similarly, area and volume judgements are based solely on the longest linear dimension. At this stage, children also rely on visual estimates to locate a point in two or three dimensional space and to reconstruct a tower of a given height. In this last task, children do demonstrate some progress from Stage I to IIA. In Stage IIA, they move the towers closer together to improve visual comparisons (Piaget calls this "manual transfer"), but comparisons are still strictly visual, and the different heights of the bases of the towers is still not accounted for.

In Stages I and IIA, children are completely unable to apply measuring instruments. In the spontaneous situation described above of reconstructing the tower, no attempt is made to use measuring instruments. In more structured situations, like those in which they are given units to measure a length, some children simply run the unit along the line, making no subdivisions into equal units. Others only cover part of the line or partition it into unequal sections. Thus, they demonstrate no understanding of a complete covering with a constant unit of measure. Since they lack conservation of the moving middle term, there is no transitivity and hence no operational concept of measurement. This failure to understand the concept of subdivision into equal units is further illustrated by Stage I children's difficulty in dividing a cake into two halves or three thirds. The youngest children forget that a given fraction implies a definite number of parts and cut the cake into any number of parts. Other children do not realize that the partition must exhaust the cake and simply cut out two pieces and leave the rest. Furthermore, they do not recognize that the sum of the fractional parts must equal the original whole.

Substage IIB. In Substage IIB (about 6 to 7) conservation is dimly perceived, and there is the beginning of transitivity so that some measurement is possible. In constructing a tower of a given height, children begin to use a moveable middle term. But instead of using one of the reliable measuring devices available, children use their own bodies, measuring with the span of their arms or with reference points on their bodies like the height of their shoulders (Piaget calls this "body transfer"). By trial and error, children gradually discover that if it takes more units to cover A than B, then A is greater than B. However, children fail to understand the importance of the size of the unit and often count a fraction of a unit as a whole or equate two quantities that measure the same number of units with different sized units of measure. Thus, "children of level IIB gradually come to make a number of true judgments, but then success is the product of intuitive adjustments and so is lacking in generality" (Piaget et al., 1960, p. 274). In attempting to locate a point in two or three dimensions, children begin to measure but only make a single measurement; they still fail to recognize the necessity of coordinating the measure of all dimensions.

Substage IIIA. The distinguishing characteristic of Stage III is the onset of operational conservation and transitivity. In Substage IIIA (7-8 years) the child conserves length and interior area and volume but not complementary area or occupied volume. In other words, they recognize the equality of areas or volumes contained within certain boundaries but do not realize that the complementary area or occupied volume (the amount of space occupied by the object in relation to other objects around it) must also be equal.

Stage IIIA children can apply a moving middle term transitively but only if it is as large as or larger than the original. Thus, they can accurately reconstruct a tower of a given height by marking off the height on a stick longer than the tower is tall but cannot reconstruct the tower measurement with a smaller unit. Operational measurement requires the synthesis of change of position and subdivision. In this stage, children possess both individually. They conserve and thus recognize that change of position does not alter quantities. Similarly, they understand composition, that the whole is the sum of the parts and is greater than any of the parts. However, these relations are qualitative, derived from the part-whole relation not from the relation of one part to another. Thus, Stage IIIA children can conserve and therefore are capable of comparing units of measure. They also recognize that a quantity is the sum of its unit covering; however, these ideas have not been fused, and measurement is not yet operational. In Stage IIIA, children continue to ignore the size and completeness of units of measure.

Substage IIIB. In Substage IIIB (8-10 yers), change of position and subdivision are coordinated and measurement through unit iteration is possible. At this stage, children can measure lengths or areas by successively applying units to cover the length or area. Stage IIIB children can also successfully locate a point in two or three dimensions. In Stage IIIA, they begin to recognize the need for two or three linear measures, but a great deal of trial and error is required. In Stage IIIB, calculation of the necessary measures is immediate.

What is lacking at Stage IIIB is the ability to apply this multidimensional awareness to calculate areas and volumes from the respective linear dimensions. This results in an interesting dichotomy. Up to this point, the development of linear, area, and volume measurement concepts have occurred concurrently. Even at this stage the concept of a unit covering is applied equally to segments and areas. However, in Stage IIIB, conservation is generalized to cover complementary area but not the parallel concept of occupied volume. Piaget's explanation for this is that the concepts of complementary area and occupied volume are dependent on an operational understanding of measurement. At Stage IIIB, measurement exists only when congruence can be established directly, i.e., only when the unit covering can be physically applied. Area can be measured directly by successive coverings of unit squares, but volume cannot because the interior is inaccessible. Therefore, the concept of complementary area develops before the concept of occupied volume.

Stage IV. Finally in Stage IV (beginning at 11-12 years) with the onset of formal operations, the development of measurement is complete. Now children calculate areas and volumes from the respective linear dimensions and conserve occupied volume. In Stage IIIB, children possess many of the prerequisites for area and volume calculations. They understand measurement through unit iteration. They can coordinate linear measurements in several dimensions to locate a point in space, and they are capable of the necessary arithmetical computations. What they lack, however, is the notion that space consists of an infinite and continuous set of points. To calculate an area, children must reduce the area to an infinite set of lines infinitely close together. Until the stage of formal operations (Stage IV) children still believe that space is comprised of a finite number of elements. For this reason, area and volume calculations are not possible until Stage IV, when concepts of infinity and continuity develop.

Discussion

It is interesting to note a trend that occurs throughout the development of measurement concepts. In the transitional stage before a concept appears in a completely operational form, it is frequently approximated in a preoperational form. For example, at Stage IIB, a type of transitivity in the form of body transfer precedes operational transitivity. Conservation also occurs on a trial and error basis at this stage. It also seems that requisite concepts may appear individually before they are fused into operational measurement. For example, a child understands both superposition and change of position individually at Substage IIIA but cannot integrate them to achieve the notion of unit iteration until Stage IIIB.

The above summary has frequently referred to the development of "operational" concepts of measurement. For Piaget, the operation of measurement is distinct from the skills of measuring. For measurement to be operational, the overt actions of the measurement process must be internalized into cognitive actions that are an integral part of a definite organized structure. For instance, an operational concept of the measurement of length cannot exist in isolation, apart from related measurement and number concepts. The development of measurement of length is mutually dependent on these related concepts and cannot be learned in an operational sense without the concurrent development of these concepts. Measurement operations must be generalized and reversible. The components of the process can be analyzed and synthesized so that a child can assess the consequence of various alterations in procedures. Thus, operational measurement requires not only that a child can apply a proper sequence of steps to measure, but he must be able to do so without slavishly following ritualized measuring procedures.

Studies of Premeasurement Concepts

Piaget's description of the development of premeasurement concepts has been the subject of a wide variety of studies which attempt to replicate, disprove, extend, or explain his conclusions. These studies cover the development of the major concepts of conservation and transitivity and deal with three basic issues: (a) validating the existence of individual cognitive operations and describing the stages of their development, (b) validating or establishing the relation between different cognitive skills, and (c) identifying the nature of the transitions between stages of development. The first issue has led to a number of replications or studies that have systematically varied some of Piaget's procedures and materials. The second has been attacked by studies in which a series of different tasks are administered to the same sample of children to establish a developmental hierarchy among them. The third has generally been dealt with through training studies. A fourth type of study that potentially can deal with all three questions involves the application of information processing techniques to describe and simulate Piagetian structures.

Research Methodology

Methodological variations create an almost overwhelming obstacle in attempting to draw consistent conclusions from the results of a great variety of studies. Differences in the criteria for success, the use of verbal or nonverbal procedures, presence or absence of conflict, material or protocol variations, and population differences make it practically impossible to equate individual studies. One result is that it is all but impossible to establish reliable age norms for the emergence of a given operation. There are also some serious problems in identifying the sequence of acquisition of different logical operations and in evaluating the effect of training.

Conservation

Research on conservation and its theoretical implications has been discussed in detail elsewhere (Beilin, 1969, 1971; Brainerd & Allen, 1971; Wallach, 1963; Wallach, 1969), so it will be treated summarily in this paper. Piaget's description of the development of conservation has generally been confirmed using a great variety of experimental procedures, materials, and types of transformations. Although the mode of assessment significantly affects the level of performance (George, 1970; King, 1971; Sawada & Nelson, 1967; Shantz & Smock, 1966; Stone, 1972; Uzgiris, 1964), the general pattern of development of conservation appears to be consistent across a great variety of stimulus situations. Examples include perceptual illusions (Murray, 1965), numerical or perceptual distractors (Carpenter, 1975), the presence or absence of transformations (Beilin, 1964), or the use of pictures instead of physical materials (Murray, 1970; Shantz & Smock, 1966).

From the perspective of the development of measurement concepts, one of the more interesting conservation studies has been reported by Beilin (1964). Using a device called the Visual Pattern Board, which displays a screen laid off in a 12 x 12 matrix of 2 inch squares, Beilin presented children from kindergarten through fourth-grade with different pairs of figures. Although children had no difficulty recognizing that two patterns with identical configuration had the same area and that a matching pattern with one square added or subtracted was not equal in area to the standard, they had significant difficulty in recognizing that two patterns with different configurations had equal areas. Beilin concluded that these errors could not be attributed to language difficulties since almost all children demonstrated an understanding of "equality" and "inequality" when presented with identical figures or figures actually differing in area. This conclusion is of questionable validity since children who incorrectly equated the terms "area" and "congruence" would give the same set of responses. However, this study does provide an example of how typical conservation errors generalize to certain measurement situations.

The sequence of development of conservation operations. One of the more intriguing aspects of Piaget's theory is the developmental hierarchy of tasks involving common operational structures. It would seem that operations with the same logical structure would transfer quite readily from one situation to another. However, this is not the case. Piaget and Inhelder (1941) propose a developmental hierarchy in which conservation of mass is attained at ages 7-8, conservation of weight at ages 9-10 and the conservation of volume at ages 11-12. This décalage has been substantiated by a number of subsequent investigations (Bat-Haee, 1971; Elkind, 1961; Hermeier, 1968; Uzgiris, 1964).

Length and area concepts, on the other hand, purportedly develop simultaneously (Piaget et al., 1960). In this case, the supporting evidence is not as clear. The results of a study by Kosanovich (1972) tend to support this conclusion, but Beilin and Franklin (1962) and Lovell, Healey, and Rowland (1962) found the development of length concepts preceded the development of area concepts. At this point, one would have to conclude that the results are equivocal. Furthermore, it may be very difficult to validate the length-area hierarchy or lack thereof. In the case of mass, weight, and volume, the age spans are substantial enough to make the relative performance of individual items fairly stable in spite of possible experimental variation. It appears that, even if there is some sort of hierarchy for length and area, it is not of this magnitude. Since it is exceptionally difficult to generate identical transformations for length and area, any results in this area may be contaminated by stimulus variation making it difficult to identify the relative effect of length and area concepts.

Training Studies. Another fundamental problem in cognitive development is to identify the nature of transition from one stage of development to the next. Piaget has attempted to describe a general theory of cognitive development which accounted for transitions between stages, but most of his early research focused on individual children performing individual tasks and provided little empirical support for his theories about the nature of the developmental process. Many of the recent studies have focused on the acquisition of cognitive structures by considering the effect of different training procedures. For a more detailed discussion of this literature, the reader is referred to reviews by Beilin (1969, 1971), Brainerd and Allen (1971), and Wallach (1969).

Beilin (1971) identifies three generations of training research. In the first generation, studies were designed to substantiate basic components of Piaget's equilibration model. One group of these studies attempted to induce conservation through cognitive conflict using either deformation or addition/subtraction (see for example Beilin, 1965; Smedslund, 1961; Smith, 1968). Other studies based on Piagetian theory relied on reversibility as the major training device (Brison, 1966; Murray, 1968; Smith, 1968).

The second generation of training studies were based on the hypothesis that Piaget's stage theory is overtly rigid in the limitations it places on cognitive development. A number of these investigators believe that the acquisition of logical structures can be accelerated and reject the equilibration model as the sole explanation for their acquisition. They do not accept, for example, that reversibility and compensation are the essential mechanisms leading to conservation. Instead, these studies focus on training children to attend to relevant attributes and disregard or ignore misleading perceptual cues (Gelman, 1969; Kingsley & Hall, 1967; Miller, 1973; Romberg & Gilbert, 1972). Other second generation studies used verbal rules (Overbeck & Schwartz, 1970; Smith, 1968) while others used a multiplicity of techniques (Owens, 1975; Steffe & Carey, 1972).

One of the more interesting studies in the learning set framework that has some direct implications for the study of measurement is reported by Bearison (1969). He used measurement procedures, counting the number of beakers containing two quantities, to significantly improve performance on liquid conservation tasks. Furthermore, the effects of training transfered to area, mass, quantity, number, and length and were maintained over a seven month period.

Studies of this second type continue to be a major force in Piagetian research. However, there is a third generation of studies whose objectives are different from the other two. The aim of these studies, which are conducted by the Genevans themselves, is to investigate the psychological mechanisms that underlie the transitions between stages (Inhelder, 1972; Inhelder & Sinclair, 1969).

Based on his review of current training research on logical operations Beilin (1971) reached the following conclusions:

> In sum, what these newer studies lead the Genevans to conclude is that the development of operativity is malleable only within the limits imposed by the nature of development. They assert quite vigorously that preoperative children do not acquire true operativity even with training. Although learning may accelerate development, acceleration is limited by the conditions of assimilation and children assimilate less of this learning in earlier stages of development. Although the possession of an elementary invariant (e.g., conservation of number) is a prerequisite to success at more advanced operative stages, the possession of a structure in one field does not lead easily

> to the acquisition of another. In fact, it may not lead to progress in another field at all. Further, language is but an instrument, and learning capacity is not provided by that instrument. Learning then, is subordinate to the laws of development, which itself follows laws that are both logical and biological. (Beilin, 1971, p. 101)

Although contrary evidence exists, virtually every type of training procedure has been able to accelerate the acquisition of conservation. However, although training can accelerate performance for a given operation, the specific mechanisms that lead to conservation are still not known. Wallach (1969) argues that such Piagetian notions as reversibility and compensation do not sufficiently explain the development of conservation, and the research reviewed by Beilin does not substantiate the conclusion that a child must be active in conflict creating situations.

One unifying view is offered by Brainerd and Allen (1971). Based on the studies that they reviewed, they concluded that the distinguishing characteristic of successful training studies was that reversibility was inherent in all their training procedures and was absent in all the unsuccessful studies. They maintain that even those studies that do not specifically train for reversibility (e.g. Gelman, 1969) actually demonstrate either overtly or covertly the inverses of specified operations. This is an interesting observation, and one that would give some structure to the divergent set of training studies. But more investigation is needed into the specific role of reversibility before this hypothesis can be given much weight. Although it is possible to identify the reversible operations in the Gelman study, one has to stretch the point to believe that they are the factor that is responsible for the development of conservation. Reversibility also does not seem to be the central mechanism in the successful Bearison (1969) study or the study reported by Inhelder and Sinclair (1969) and reversibility does not account for successful training of transitivity (this is discussed later). In any case, it is an interesting hypothesis that warrants further systematic investigation. But at this stage it is probably best to stick to Beilin's conclusion that there is little evidence for the preeminence of any single set of training procedures.

Another of Beilin's (1971) conclusions is that virtually no method is effective with very young children. This leads to the hypothesis that training does not create new or different logical operations but simply extends the domain of already existing operations. No learning is likely to occur if the concepts to be learned are outside the operational domain of the children. Thus, conservation training does not make conservers, it simply teaches them to extend their knowledge of conservation to the testing situation. In the terminology of the Genevans, learning only accelerates development; it does not initiate it.

Virtually all training studies have found that trained conservers can transfer their knowledge of conservation to novel materials that were not used in the training procedure (specified transfer). Nonspecific transfer (e.g., transfer of training on length to area) has been more difficult to achieve, although examples do exist (Bearison, 1969; Gelman, 1969). In spite of these few exceptions, there seems to be relatively little transfer between Piagetian operations, both for trained conservers and for natural conservers. Why operations do not generalize more readily to different perceptual situations is one of the puzzling aspects of Piagetian research, especially given the preeminence of the logical operations in the theory.

Transitivity

Transitivity studies have been even more vulnerable to methological variations then conservation studies. As a consequence the age for the development of the concept of transitivity has been placed as early as 4 years (Bailey, 1971) and as late as 8 years (Smedslund, 1963b).

Attempts to eliminate possible task ambiguity by pretraining (Bailey, 1971) or using nonverbal or operationally defined assessment techniques (Braine, 1964) have generally identified an earlier development of transitivity than traditional studies that relied on verbal techniques (Piaget et al., 1960; Smedslund, 1963a, 1963b, 1964). It has also been found that providing memory aids significantly increases the number of transitive inferences (Roodin & Gruen, 1970), which possibly implies that some children who are capable of making transitive inferences fail to do so because they forget the relevant quantitative relations. Research by Bryant and Trabasso (1971) also indicates that failure on transitivity tasks may result from forgetting the premises upon which the transitive inferences is based. An extension of this study by Riley and Trabasso (1974) indicates that learning both A is greater than B and B is less than A significantly increases retention of the relations upon which the transitive inferences are based and therefore increases performance on transitive tasks. However, the hypothesis was raised that subjects in these studies may attain a solution without using a logical transitive inference by encoding the information into an ordered spatial array and solving the problem by internally scanning the array.

A classic debate between Braine (1959, 1964) and Smedslund (1963b, 1965) revolved around assessment techniques and exactly what constitutes evidence of transitivity. In general, many of the studies which have identified the earlier development of transitivity are vulnerable to charges of accepting pseudotransitive judgements. On the other hand transitivity failures in studies of Smedslund (1963a, 1963b, 1964) or Youniss and Murray (1970) possibly result from specific task ambiguity or complexity.

Although assessment issues have not been resolved, it appears that a major source of variation results from the presence or absence of conflicting perceptual cues. Many children who seem to correctly apply the transitive property of length in a neutral situation where there is no conflict fail to do so in the presence of a Muller-Lyer illusion (Divers, 1972; Jones, 1969; Trenary, 1972). The evidence seems to suggest that the ability to apply the transitive property exists at various levels. A child first learns to apply the transitive property in simple situations without conflict but is not immediately able to apply it in more complex situations or situations involving conflict.

Some caution must be exercised in assessing transitivity in the absence of perceptual conflict. In order to be classified as possessing the transitive property, it seems reasonable that a child should be able to identify situations where the transitive property does not apply (e.g., $a < b$, $b > c$) as well as apply it when it does. Bailey (1971) found that younger children have more difficulty identifying nontransitive situations than in making transitive inferences. This seems to indicate that some sort of pseudotransitivity is a significant danger in the absence of perceptual conflict, and experiments should guard against it by including a variety of transitive and nontransitive tasks in assessing the presence of transitivity.

Conservation-transitivity. Identifying the sequence of development of conservation and transitive operations seems to be a function of how transitivity is defined. Piaget and Inhelder (1941) proposed that conservation and transitivity develop synchronously. However, with the exception of a study by Lovell and Ogilvie (1961b) most if the initial replications found that conservation develops before transitivity (Koolstra, 1964; McManis, 1969; Smedslund, 1961, 1963a, 1964; Steffe & Carey, 1972). These studies, however, have been criticized by Brainerd (1973a) for failing to equate the relative sensitivities of the assessment tasks. Each of the studies employed perceptual illusion in the transitivity tasks. Using tasks that did not involve perceptual illusion, Brainerd (1973b) found that the development of transitivity precedes the development of conservation. It is not clear that Brainerd's procedures are any more equitable than the others, since the conservation tasks involved perceptual illusion while the transitive did not. At this point, the most reasonable conclusion seems to be that the sequence of development appears to depend on what evidence one requires for the respective operations. If one compares the standard conservation tasks to the weaker definition of transitivity, then it appears that transitivity develops earlier. If one insists on stronger criteria for transitivity, then it appears that conservation develops earlier.

Training studies. The results of transitivity training studies do not depart significantly from those of conservation training studies. Brainerd (1974), Owens (1975), and Smedslund (1963a) induced gains in transitivity through training. In fact, Brainerd (1974) and Owens (1975) both found that for a given sample of children training in transitivity was more successful than training in conservation. These results conflict with the results of a study by Garcez (1969), who found that no subjects trained in transitivity alone attained operational levels while 28 percent trained in conservation alone and 24 percent of those trained in both gave operational responses for the trained operations.

As with conservation, success in achieving transfer to new operations has been limited. Brainerd (1974) found that training in transitivity of length transferred to transitivity of weight. However, neither he nor Garcez found significant transfer from transitivity training to conservation tasks or vice versa. Steffe and Carey (1972) also found no significant transfer between conservation training and performance on transitivity tasks, and Johnson (1974) found no evidence of transfer from classification and seriation training to either conservation or transitivity.

Only one study (Inhelder & Sinclair, 1969) reports success in achieving transfer between different logical operations. They trained children to conserve weight by confronting them with the discrepancy between their predictions and the actual outcomes of weighing and by asking them to establish the relative weights of different objects. Eighty-six percent acquired conservation of weight, and 64 percent of those who acquired conservation were capable of performing transitive operations. There is one significant factor in this study that the Genevans conclude is responsible for their success. All children started from a true operational level in that they were able to conserve liquid quantity. Thus, the training accelerated the generalization of established operations but did not initiate the development of new operations.

Until this study has been replicated, these results should be regarded with some caution. The Genevans generally use a small, select sample of subjects and do not exert rigorous experimental control. In this study, they failed to pretest for transitivity. If transitivity preceeds conservation, as hypothesized by Brainerd, then many of the subjects in the study may have attained transitivity prior to instruction and therefore did not learn it as a result of their conservation training. It is also possible that their training included some practice in transitivity. It would be premature to conclude that conservation training readily transfers to transitivity, even with children who start the training at established operational levels. On the other hand, the Genevan procedures involved more intensive individualized training than most training studies have employed. It is possible that the reason for the general lack of success in achieving transfer is that subjects in most studies have not been trained to a genuine operational level.

Information Processing

A recent development in the study of cognitive development has been the application of information processing techniques to describe and explain Piagetian operations (Baylor, Gascon, Lemoyne, & Pothier, 1973; Klahr & Wallace, 1970). Instead of analyzing behavior in terms of the logical and algebraic properties of the problem, the approach is to analyze the information processing requirements of the task. In other words, behavior is described in terms of the subroutines a child would need to apply in order to perform a given task, which is analogous to the compilation and execution of a computer. This involves (a) encoding external stimuli, (b) assembling task specific routines from a repertoire of fundamental processes, and (c) executing the task specific routines. This not only forces the programmer to develop an explicit description of the behaviors involved in each task, but some sort of analysis on the demand of subroutines may provide an indication of the level of difficulty of each task.

Baylor et al. (1973) applied these techniques to analyze the problem of seriating weights. They videotaped three children in different stages of development of seriation concepts. A detailed protocol analysis of their responses was compiled, and a computer was programmed to simulate their behavior. They concluded that for the seriation task intellectual development is related to (a) progressive sophistication in structuring the environment, (b) better use of memory, (c) span for drawing inferences, and (d) initial conception of what seriation is.

To test the feasibility of an information processing approach, Baylor and his associates conducted a second study. By holding the information processing requirements of the tasks isomorphic, they found that the well known décalage between seriation of length and seriation of weight disappears. They interpreted these findings as support for the validity of an information processing approach to the analysis of cognitive development.

Measurement Stage

The literature on the development of numerical measurement processes is not as abundant as the literature in premeasurement processes, and studies of how children learn to calculate with area and volume formulas are virtually nonexistent. Many measurement tasks have never been replicated outside Geneva. Since Piaget himself admits that the results of some measurement tasks are influenced by school curricula, one must accept his description of the advanced stages of the development of measurement concepts with some caution. This is especially true given the Genevan's lack of rigorous experimental control, their small samples of subjects, and their tendency to extrapolate extensively on the basis of limited empirical evidence.

There are two major groups of studies dealing with the development of measurement concepts. First, there are studies that attempt to validate Piaget's proposed hierarchy. Second, there is a variety of studies dealing with children's understanding of the concept of a unit of measure.

Validation Studies

Four studies have attempted to validate Piaget's proposed hierarchy of measurement concepts by administering a collection of measurement and measurement related tasks to the same set of children (Andrejczak, 1972; Lovell, Healey, & Rowland, 1962; Needleman, 1970; Wagman, 1975). Although each study identified some minor variations, they generally found no radical departure from Piaget's proposed sequence of development. These studies, as well as studies by Lovell and Ogilvie (1961a) and Lunzer (1960), also confirmed that Stage IV operations are not attained until at least 11 or 12 years of age.

One of Piaget's conclusions that has not stood up under further investigation involves the relation of spatial and measurement concepts. As noted above, Piaget believes that area and volume calculations are not possible until children recognize that space consists of an infinite and continuous set of points. This hypothesis was not supported by the studies of Andrejczak (1972) and Lunzer (1960).

The Concept of a Unit of Measure

The unique feature of the measurement process that distinguishes it from simply counting is the unit of measure. In assigning a number to a set, the units are the individual elements of the set. However, in the measurement process the individual units that are counted may not be distinguishable, and different units may be used to measure the same quantity. This second feature of units of measure has been the subject of a variety of studies. There are three studies that describe the difficulties children encounter in dealing with different units and three excellent training studies involving measuring unit concepts.

The first study (Carpenter, 1975) employed a series of conservation and measurement tasks in which children were provided both measurement and visual cues regarding the relationship between two liquid quantities. In some tasks children had to focus on the visual cues; the liquid was in identical containers and was measured with different units. In others the same unit was used, so children had to focus on the numerical cues since the visual cues were misleading.

This study found that, contrary to earlier hypotheses, virtually all first- and second-grade children respond to numerical measurement cues at least as readily as to perceptual cues. However, the majority still center on a single dominant dimension, numerical or perceptual depending on the problem situation. This leads to a number of incorrect judgements. For example, when two quantities in identical containers are measured with different units, children abandon the valid visual cues and compare the quantities strictly on the basis of the number of units each measured. It is interesting that there is no significant difference in difficulty between this problem, in which the distracting cues are numerical, and the traditional liquid conservation problem, in which the distracting cues are visual. On the other hand, problems in which the liquid is measured into different shaped containers using the same unit of measure are significantly easier than the traditional conservation problem. In other words, appropriate numerical cues are attended to even more readily than appropriate visual cues. In fact, most first- and second-graders can use appropriate measurement cues that follow distracting visual cues. This was demonstrated by the fact that almost all subjects could correctly identify the relation between quantities in different shaped containers that were subsequently measured with a single unit. These results seem to imply that first- and second-grade children do naturally attend to the results of measurement operations, and measurement operations used appropriately may facilitate conservation judgments.

Although the subjects could operate with selected measurement skills, they had considerable difficulty dealing with the compensating relationship between unit size and the number of units measured. In the problems discussed, the greatest difficulties occurred when quantities were measured with different units of measure. This type of problem was significantly more difficult when the larger unit was not visually identifiable. In other words, being able to see the compensating relationship between the unit size and the number of units was a significant factor that influenced the responses of over 10 percent of the subjects tested. By far the most difficult task was to determine the relation between two units by observing for each unit the number of units in a given quantity.

The results of this study indicate that first- and second-grade children have considerable difficulty dealing with the unit size-number of units relationship. However, a subsequent investigation (Carpenter & Lewis, in press) revealed that although they may have difficulty applying their knowledge, many first- and second-graders do recognize that a compensating relationship exists. A significant number of subjects who failed the above measurement problem in which equal quantities were measured with equal units could successfully predict that a quantity measured with the larger unit would measure fewer units than an equal quantity measured with the smaller unit. Thus, it appears that children understand that such an inverse relation exists before they are able to generalize this knowledge to situations involving direct conflict. This situation corresponds to the case of transitivity, in which it was also concluded that transitive operations do not immediately generalize to conflict situations. Since the understanding of compensation precedes the ability to deal with it in conflict situations, it is difficult to account for its acquisition through any sort of measurement experience.

Another study which tested first-, second-, and third-grade students' ability to deal with different units is reported by Bailey (1974). He administered four different tests in which subjects were asked to compare the length of two polygonal paths. In one test there was an equal number of segments in each path, but the segments in one path were longer than the segments in the other. The second test involved an unequal number of congruent segments. In the third, one path had longer but fewer segments. In the fourth, the segments in both paths were equal in number and length. The results and procedures are reported somewhat sketchily, which make it difficult to establish the criteria used to reach the given conclusions. But it appears that children had a great deal of difficulty in coordinating the number of units with the length of the units in establishing the relative length of the polygonal paths. In fact, only 3 of 90 subjects (all third-graders) used both dimensions in establishing the length relation between the two paths. The most difficult task was the third, in which no valid comparison was possible. Bailey also concluded that transitivity was a prerequisite for successful completion of the tasks.

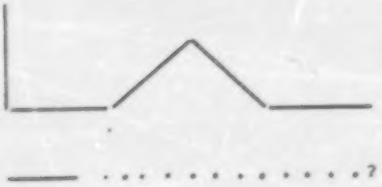
Training studies. Some of the most interesting training studies have been conducted by the Genevans. The unique feature of their training procedures is that they are conducted in a clinical setting with careful assessment of each subject's stage of development and detailed observation of the precise nature of the effects of training. Although these procedures allow a great deal of subjective judgment on the part of the experimenter, they provide some insight into the exact mechanisms of learning rather than simply observing that some learning occurred.

One particular study that has been widely reported (Inhelder, 1972; Inhelder & Sinclair, 1969; Sinclair, 1971) has profound implications for instruction in measurement operations. The learning procedure the Genevans employed involves leading a subject who has already acquired a given operational structure to develop a new structure which is normally acquired later. Since number conservation is acquired two to three years earlier than length conservation, it was hypothesized that elementary concepts of linear measurement could be facilitated by exercises in which numerical operations could be used to evaluate length.

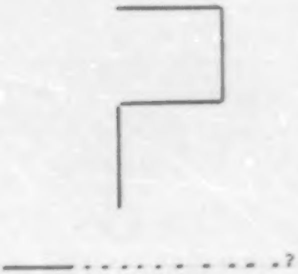
The experiment required subjects to construct lines out of matchsticks that are the same length as figures constructed by the experimenter. The difficulty results from the fact that the experimenter's matches were longer than the subjects' matches (in the ratio 7:5). Three situations were presented in the sequence illustrated in Figure 1. The three situations remained in front of the subject. After he had completed his first three solutions, he was asked to give explanations and eventually to reconsider earlier solutions

All subjects, mean age six years, had passed a pretest of numerical conservation and had failed a pretest of linear conservation. On the posttest 35 percent made no progress, 37 percent made some progress, and 28 percent gave correct answers and justifications for all items on the posttest.

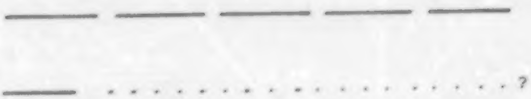
Most interesting, however, is the description of the learning process and the various stages of development. Subjects who showed no progress centered on a single dimension. If they were asked a length question or provided with length cues as in situation 1, they would center on length.



Problem 1



Problem 2



Problem 3

Figure 1. Conservation training problems.

If asked a number question, they would center on number and ignore length. There was successive application of two distinct systems, and these subjects saw no contradictions in their responses.

In the second stage evaluation schemas seemed to be present simultaneously. In this stage, subjects were not satisfied with either a purely numerical or linear solution and turned from one solution to the next. However, they were not capable of a new solution that accounted for the other two. Thus, although they were aware of the contradiction they could not resolve it.

In the third stage some attempt was made at integration, but the result was an inadequate "compromise solution." Some subjects broke a match so as to have the same number and still not have a path that went beyond. Others ignored the instructions and constructed a nonlinear path.

In the fourth stage the different schemas were integrated and coherence was attained. Subjects recognized that you need more matches when they are smaller and could use the results of situation 3 to correct the solutions in situations 1 and 2.

Thus, the application of already acquired numerical operations can be used to facilitate the acquisition of spatial measurement. But complete acquisition is a very long and difficult process, even with intensive clinical training. The Genevans have concluded that the conflict resulting from the misinterpretation of misleading cues is the mechanism that leads to operational measurement. Furthermore, it is the subject's active effort to discover compensatory and coordinating actions and not the visual results of the experiment that leads to higher order structures.

Another study that investigated how the concept of a unit affects children's learning of measurement concepts is reported by Montgomery (1973). This study was an aptitude-treatment interaction study which examined the interaction of second- and third-grader's ability to learn unit of length concepts with two treatments based on area and unit of area concepts. Aptitude was measured using a teach-test procedure which partitioned subjects on their ability to learn to compare two lengths measured with different units. Subjects were randomly assigned to one of two nine-day instructional treatments on measuring and comparing areas. The difference between the treatments was the emphasis placed on the unit of measure. In one treatment, subjects always measured with congruent units and compared regions covered with congruent units. In the other treatment, subjects measured with noncongruent units and compared regions covered with different units. On both a posttest and a retention test, the treatment that used different units was significantly more successful in teaching children to assign a number to a region (measure) and to compare two regions using their measures. However, there was no significant difference between the two treatments on a transfer test that included problems involving measurement with different units, and no significant interactions were found between aptitude levels and treatments.

These results together with those reported by Inhelder and Sinclair (1969) imply that conflict induced through measuring with different units is a facilitating mechanism in learning measurement concepts. Montgomery's study seems to imply that even children who do not readily grasp the interaction of the unit size and the number units measured benefit from the conflict that the use of different units induces. Using multiple units may not lead to an understanding of the unit size-number of units relationship, but it seems to facilitate the development of other measurement concepts.

The Genevans studied unit concepts in a clinical setting. Montgomery used a more traditional classroom instructional setting for her training session. Gal'perin and Georgiev (1969) report the results of a study in which the entire kindergarten mathematics curriculum was based on the concept of a unit of measure. Their hypothesis was that the traditional emphasis on number concepts incorrectly characterizes units as discrete entities. By introducing number as a property of sets, the traditional curriculum induces an orientation that leads to a number of basic misconceptions.

To test their hypothesis, they administered a series of measurement problems to the "upper group" of a Soviet kindergarten. They concluded that young children who are taught by traditional methods lack a basic understanding of a unit of measure. They do not recognize that each unit may not be directly identifiable as an entity and that the unit itself may consist of parts. They are indifferent to the size and fullness of a unit of measure and have more faith in direct visual comparison of quantities than in measurement by a given unit. This same set of items has been administered to a group of American first-graders (Carpenter, 1971). Although some conclusions were modified based on an additional set of items and differences in interpretation, the general results were confirmed for American first-graders.

On the basis of this study, Gal'perin and Georgiev devised a program of 68 lessons that focused on measurement concepts and systematically differentiated between units of measure and separate entities. The lessons were divided into three parts. The first part dealt with forming a mathematical approach to the study of quantities. This section focused on replacing the habit of direct visual comparison with systematic application of measuring units. Appropriate units for measuring different quantities were identified and measuring skills were studied directly, with special attention directed to the deficiences identified in the pretest. A variety of units was used, including units consisting of several parts (two or three matches, spoons, etc.) or some fractions of a larger object (half a mug or stick). All of these concepts were presented without assigning numbers to the quantities.

It was not until the second part that the concept of number was introduced. Thus, Gal'perin and Georgiev introduced most of the basic measuring skills and spatial concepts before they introduced numbers.

Note that this sequence completely reverses that described by Inhelder and Sinclair. In the third part, the inverse relationship between the size of the unit and the number of units was introduced.

Although the investigation was not conducted with strict experimental controls, the students who participated in this program showed striking gains over the performance of the previous year's students. Whereas fewer than half the students in the previous year could answer most of the items on the measurement test, performance was close to 100 percent for the experimental group. Since experimental and instructional procedures are described only briefly, it is difficult to document the exact nature or cause of the gains. It may be that the treatment simply sensitized the children to what the experimenter was looking for in the measurement test. On the other hand, the gains are so striking and involve such fundamental concepts that this study is worth serious consideration.

Surveys of Measurement Knowledge and Skills

Earlier in this paper a distinction was made between understanding measurement concepts and familiarity with measuring skills. The research that has been surveyed to this point has dealt with understanding basic measurement concepts. In addition to this research, there are a number of studies that surveyed children's familiarity with standard measuring units or terminology (Davis, 1959; MacLatchy, 1950, 1951; McKnight, 1965; Mermelstein, 1964; Murphy, 1969, Spayde, 1953) or their skill in estimating or measuring (Corle, 1960; Scott, 1966; Wilson & Cassell, 1953). Most of these studies are too dated to be of much value or lack generalizability because they were conducted with a narrow cross section of the population.

The best single source of children's and adult's familiarity with measurement terminology and skill in measuring is the results of The National Assessment of Educational Progress (National Assessment of Educational Progress, in press). In their survey of mathematics skills, National Assessment included a number of measurement exercises. These items included problems in converting and comparing standard units of measure, estimating and measuring length, and calculations of perimeter, area, and volume. Since these exercises were administered to carefully selected groups of nine-year-olds, thirteen-year-olds, seventeen-year-olds, and young adults, they provide a representative picture of American's familiarity with measurement skills and terminology. However, it is a somewhat incomplete picture, as many basic measuring skills were not tested.

The results of the National Assessment measurement exercises indicate that all age groups except nine-year-olds can compare quantities measured with different standard units of measure, and even nine-year-olds are generally successful with comparisons involving feet and yards or pints and quarts. However, specific conversions are significantly more difficult for all age levels. It is also worth noting that only about half of the seventeen-year-olds or young adults demonstrate any familiarity with standard metric units.

Most nine-and thirteen-year-olds are able to make simple linear measurements. However, measurement operations involving fractions of a unit or requiring that the ruler be moved are significantly more difficult, as are problems involving indirect measurement. Nine-year-olds are much more affected by these factors than thirteen-year-olds.

At all levels for virtually every exercise, perimeter, area, and volume problems are exceptionally difficult. Fewer than half the nine-year-olds can successfully compare the areas of rectangles that are subdivided into unit squares. Less than a quarter of the thirteen-year-olds and half of the seventeen-year-olds can calculate the volume of a simple rectangular parallelepiped subdivided into unit cubes. Problem situations involving area calculation are even more difficult. The results for the nine-year-olds generally are consistent with Piaget's conclusion that the notion of a unit covering appears in Stage IIIB at about the age of 9-10 years. But, the volume problem indicates that the operations of Stage IV do not develop for a substantial majority of the population by age 13 and are not even learned by most of the seventeen-year-olds.

State assessment reports provide another source of baseline data on children's ability to apply measurement concepts. A number of states--including Florida, Michigan, Texas, and Wisconsin--now use criterion referenced testing procedures. These results provide a valuable supplement to the National Assessment results. On the whole their results support the conclusions drawn from the National Assessment results. Taken together the results from National Assessment and the state assessments indicate that, even in the upper grades, a substantial number of children have not learned a number of fundamental measurement concepts.

Instructional Studies

A number of studies have investigated the effectiveness of various instructional strategies in teaching measurement skills. In general, these studies have lacked any unifying conceptual basis, and it is difficult to assess their contribution. Several studies (Williams, 1970, Young, 1969) have claimed success in teaching measurement concepts to three- and four-year-olds, but they provide no evidence to indicate that anything more than rote learning occurred. On the other hand Luchins and Luchins (1947) and Wertheimer (1945) were specifically concerned with meaningful learning and report examples of children as young as five or six generalizing the rectangle formula to discover a method for calculating the area of a parallelogram. Although they make no claims that they are reporting typical responses for this age, these results are exceptional in that Piaget's developmental stages would not include these skills until the stage of formal operations at 11 or 12.

A number of other studies have investigated the effectiveness of different instructional strategies in teaching different measurement concepts (Bargmann, 1973; Craig, 1973; Eroh, 1967; Johnson, 1970; McFee, 1968; Mueller, 1969; Pattison, 1973; Richards, 1971; Urbach, 1973). On the whole, they provide no conclusive evidence for the superiority of any specific instructional strategy in teaching measurement concepts.

Most mathematics programs have ignored Piaget's description of the development of measurement concepts and other cognitive processes (see, for example, Huntington, 1970) and have dealt with measurement at a skill level. However, several programs have been influenced by learning research and provide specific instruction in fundamental measurement processes (e.g., AAAS Science: A Process Approach, Developing Mathematics Processes (DMP), and Science Curriculum Improvement Study (SCIS)). By investigating the relative effectiveness of these existing programs, one might get some insight into the long range effects of instruction that cannot be gained from short training studies. Two such studies exist. Kamps (1971) investigated effectiveness of three second-grade programs (AAAS, a program built around Cuisenaire Rods, and a program which included little measurement) in teaching six conservation and measurement concepts taken from The Child's Conception of Geometry (Piaget et al., 1960). He found that AAAS students scored higher on conservation tasks, but there was no difference in overall achievement between the three programs.

Almy and associates (Almy, Dimtrovsky, Hardeman, Gordis, Chittenden, & Elliot, 1970), in a longitudinal study of logical thinking in the second-grade, compared programs using AAAS together with the Greater Cleveland Mathematics Project (GCMP) materials, SCIS together with GCMP alone with no prescribed science program, and a program with no prescribed lessons in either science or mathematics. She found some evidence that the children using only the GCMP program scored lower on a conservation of weight task. However, she found much greater differences for a transitivity of length task. In this case, the AAAS group scored the highest followed by the group with no prescribed lessons, with the GCMP and GCMP/SCIS groups about the same. Almy also found that second-grade children in schools in which systematic instruction in mathematics and science was initiated in kindergarten scored higher on the conservation task than children who did not begin systematic mathematics and science instruction until the first-grade.

In another study investigating the relative effectiveness of different curriculum programs in teaching measurement concepts, Friebel (1967) found that sixth- and seventh-graders using the SMSG program scored significantly higher on a test of measurement understanding than corresponding groups instructed with traditional programs. The results of all three of these studies must be interpreted with some caution as it was not possible to randomly assign subjects to treatments or to maintain strict experimental controls in classroom settings involving large numbers of students.

Discussion

If one accepts Piaget's criteria to establish the existence of a given operation, the research supports the conclusion that premeasurement concepts become operational before measurement concepts do. However, the research also indicates that children can and do attend to selected measurement concepts before premeasurement concepts are fully operational. In fact, the studies reported by Bearison (1969) and Inhelder and Sinclair (1969) indicate that certain measurement experiences can accelerate the acquisition of premeasurement concepts.

Proposed Models of Cognitive Development

This review of measurement research points to the conclusion that there has been an exaggerated emphasis on internalized logical-mathematical structures. Among other things, this has resulted in much needless controversy over exactly what evidence is required in order to conclude that a child has attained a given operational level. The research on measurement suggests that it is not the existence of internal logical-mathematical structures that limits performance. Children possess such structures long before they can widely apply them. No individual task or group of tasks can conclusively demonstrate the existence or absence of a give operation. At best they indicate that a child can apply a given operation to a given task. The research on transitivity and the notion of horizontal décalage indicate that task specific variables have a profound effect on performance and should be included in any equation describing cognitive development.

Flavell and Wohlwill (1969) propose that an analysis of cognitive development should incorporate a competence-performance distinction similar to Chomsky's model for language acquisition. The competence component of the model is the logical-mathematical structure of the domain, and the performance component represents the psychological processes by which the structures in the competence component get accessed to specific tasks. The competence component is an idealized abstract representation of what is known or understood, whereas the performance component must account for the reality of stimulus variations, conflicting information, memory limitations, etc.

A similar approach is the information processing model proposed by Baylor et al. (1973) and Klahr and Wallace (1970). This approach has generally been limited to situations like seriation, in which there is an abundance of observable action which can be taken to imply the application of certain strategies. It may be much more difficult to build information processing models for conservation, in which there is less overt action. There is also the difficulty that a child's logic is not always congruent with adult logic, and it may be very difficult to identify the specific heuristic strategies a child is really using or what aspect of the stimulus situation he is attending to. Nevertheless, many of the logical difficulties that occur in the development of measurement concepts seem to relate to information processing variables, especially if sufficient emphasis is given to the encoding process. Such constraints as stimulus load, demands on memory and different logical operations (subroutines), the requirement of simultaneously considering several variables, etc. are significant factors in the development of the measurement processes. All appear susceptable to information processing analysis. It is even possible that a more productive description of the development of measurement concepts would involve levels of information processing capabilities rather than stages of logical operations. In fact, the stages of learning outlined by Inhelder (1972) in describing the results of the measurement training exercise seem to be moving in this direction.

Training

It is generally accepted that appropriate training can accelerate the development of specific measurement concepts. Furthermore, training in measurement seems to accelerate rather than depend upon the development of concepts of conservation and transitivity. It might be hypothesized that the effectiveness of training is more a function of the information processing demands of the specific tasks than of the development of prerequisite logical operations. In other words, children may benefit from training as long as the information processing demands of the tasks do not exceed their limits, in spite of the fact that they do not possess the prerequisite logical operations. A child's logic is not the same as adult logic. Given appropriate instruction, they may be able to attend to certain relevant dimensions of a stimulus situation and ignore the fact that their judgments depend on certain prerequisite knowledge that they lack.

Although the research suggests that training can accelerate development, measurement learning studies have failed to identify the specific mechanisms of development. There are no critical experiments that confirm or preclude the major theoretical positions, as the results of most training studies can be explained in either behaviorist or equilibration terms.

There are also some serious shortcomings observed in the training of basic measurement concepts. One limitation is the general failure of subjects to transfer their learning to related tasks. Second, training procedures, even in the most successful studies, have failed to improve the performance of a substantial number of the subjects receiving the training. The recent training research of the Genevans suggests that only subjects who have achieved conservation of number benefit from training in logical operations.

In interpreting these results in terms of the Flavell and Wohlwill competency-performance model, it might be hypothesized that training generally affects the performance component of the model and does not affect the basic comeptencies. In other words, training may not generate new operational structures but rather trains subjects to generalize existing operations to new situations. From an information processing perspective, one might speculate that training does not result in a higher level of information processing but simply shifts the domain in which the established level can be applied.

Educational Implications

While there is an abundance of clinical research on the development of measurement concepts, and a number of recommendations as to how this research might apply to the mathematics curriculum (see for example Beilin, 1971; Lovell, 1966, 1972; Steffe, 1971), there is little direct research relating the results of clinical studies to measurement instruction in the mathematics curriculum. Although research has identified levels of development of measurement concepts and rough age approximations for the development of certain operations, it is not immediately clear what implication this has for the curriculum. Beilin (1971) proposes that the development of logical operations may depend on prior knowledge that must be learned by rote. If this is the case, fitting instruction to coincide with appropriate developmental stages might actually retard development. Some support for this position is found in the study reported by Inhelder and Sinclair, which indicated that measurement experiences accelerate the development of conservation, even though conservation is a logical prerequisite for measurement.

The clinical training procedures that have induced various operations have frequently required sustained effort to achieve very modest gains. The direct translation of the methods of these studies to the mathematics curriculum is at this point unwarranted. To paraphrase Beilin (1971), the research on measurement has shown how complex a process the growth of measurement concepts can be, but it has not demonstrated how it can be made easier.

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Needed Research

on

Teaching and Learning Measure

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Introduction

The teaching and learning of measure has been investigated by a number of different individuals using a variety of experimental procedures. The most productive studies have considered children's understanding of fundamental measurement concepts and how children learn these concepts. The development of measure concepts proposed by Piaget and his associates (Piaget & Inhelder, 1941; Piaget, Inhelder, & Szeminska, 1960) has been the focus of most of this research. Although there have been a variety of experimental procedures employed within a Piagetian framework, Piaget's theories provide some unity to this research and give the results some sense of coherence. The research that has focused strictly on how to teach measurement skills lacks any unifying conceptual basis and has contributed little to the understanding of teaching or learning of measure.

Defining a unifying conceptual basis for the learning and teaching of measure is one of the major needs of researchers. Such a unifying conceptual basis should encompass previously discovered factors affecting the learning of measure. In addition, this basis should suggest new problems for researchers and accommodate to the instructional constraints of the schools. Identifying, describing, and testing the potential of such a model is perhaps the foremost need of researchers in the field of how children cope with measure.

Although the unifying influence of Piaget's work has resulted in research that provides a fairly consistent picture of how children at different stages of development operate within a variety of fundamental measurement situations, the conceptual framework that it has provided has proved to be inadequate on two counts. First, Piagetian measurement research has focused on a relatively narrow cross section of measurement concepts and processes. Most of this research has dealt with primary notions like conservation and transitivity, and many of the more complex measurement operations have been virtually ignored. What is needed is a more comprehensive unifying basis that will focus on a broader scope of measurement research problems.

A second limitation of much Piagetian research has been its attempt to describe behavior strictly in terms of the relationship between logical-mathematical structures. This analysis has proved deficient in that it does not easily account for many observed variations in performance and is not very useful in describing the effects of different training procedures. What is needed is research that will begin to generate a more comprehensive model of cognitive development that can more adequately predict and account for the relationship between different cognitive skills.

Thus, research needs to take a more inclusive perspective both of the types of activities that are included as measurement and the range of factors that influence behavior with respect to measurement operations. In other words, we need a unifying basis that broadens the scope of measurement research and a more complete model for analyzing and describing the types of behavior that occur within individual studies.

The Homomorphic-Transfer Model

One candidate for a unifying conceptual basis is the homomorphic-transfer model described by Osborne in this volume. This model is based on the fundamental functional character of measure systems, that is, that a measure system associates an entity with a number. The entity is an element of a structured space, the domain of the function. For a single measure system the learner must acquire understanding and skills stemming from the (a) characteristics of the domain space, (b) characteristics of the range space, and (c) the linking homomorphic association of the two spaces. These three sets of learnings constitute a research problem in the transfer of learning. We label this as the within transfer problem since it is concerned with a single system of measure, such as area. Thus, the research needs to address questions concerning each type of acquisition.

Research into the matter of how children use cues from either the domain space or the range space to provide insight into the other space should be of practical significance in engineering instructional sequences for the classroom. Clearly, little is known concerning how children tie together concepts and skills of these two spaces to create the homomorphism that is a measure system. A more careful examination of the cueing mechanisms in terms of transfer may be revealing. It seems reasonable to study cueing mechanisms for several different measure systems. Studies of the cueing variable need to be conducted that recognize that measure systems differ in terms of the perceptual referents to which the child has access. Perceptually, comparison of volumes is quite different than comparison of speeds. Does this perceptual difference affect the use of cues by the child? Do numerical cues assume greater significance in the learning process as the perceptual base in the domain space becomes more complex and/or indirect?

Another factor that needs attention is the cueing variable; the shifting from a unit iteration basis for incorporating number into the measure system to a multiplication based incorporation of number. For example, for area and volume concepts, teachers begin to develop the concept of a unit using situations in which children "fill" the domain space. Then they switch to using $A = lw$ or $V = l \cdot w \cdot h$, respectively, often without sufficient attention to tying together the space-filling, unit iteration and the multiplicative ideas. Surprisingly little research has explored the effects of this switch and how it can be expeditiously accomplished. Does the experience with the multiplication torpedo the child's understanding of unit and iteration or, to say it another way, does it provide an interference with the unit ideas and the premeasure domain space ideas? It may well be that memory is the operant factor for children whose concept of the space-filling character of units is incomplete. Young children tend to concentrate on positional and discrete characteristics of units in many early measure situations. Do these children need special experiences to help them establish the multiplicative version of measure?

Perception is necessarily a factor in measure. The child encounters approximation and error in coping with measurement. For example, this happens when a child uses a ruler and the end of the object being measured does not neatly align with a mark on the ruler or when a polygon cannot be exactly covered with units of area. For the young child, instructional materials are designed to be "nice" or to "fit." Then at some later point, conflict is introduced. Finally, relative error, accuracy, and other related concepts are introduced. But we have no research-based evidence concerning how and when these concepts may be appropriately introduced nor the effect of these inescapable perceptual conflicts on the child's learning about area. Most of the measure related instructional materials appear to build in a perceptual regularity and nicety (problems come out right) that is missing from real world measurement. Several creative articles that address these types of instructional problems (Payne & Seber, 1959; Trimble, 1974) have advocated seemingly sensible approaches, but none are based on research evidence. We simply do not know how these factors of perceptual reality affect children's understanding and learning of measure systems.

Structural Transfer and Measure Learning

A measure system is a complex structure of ideas. Pairs of measure systems share many of these ideas. Previous research has explored how children form concepts and skills within a single system of measure rather than purposefully looking for instructional advantage in building concepts for one measure system in terms of pervious learning in another measure system. Within the unifying conceptual basis advocated in this paper, the homomorphic-transfer model of measure learning, this sort of learning is described as the across transfer learning problem as defined in the paper by Osborne in this volume.

The across transfer question characteristic of the homomorphic-transfer model will necessarily address some outstanding needs of teachers and curriculum designers as they work with measure and measurement. Across transfer questions may also focus attention on older as well as younger learners. Necessarily, the across transfer setting demands that subjects have some previous learning in a measure system to set up the potential for transfer. This need for prior learning and experience with measure will tend to shift the average age of subjects upward. This is not inappropriate; the majority of school instruction directed to measure is with older children at the upper elementary or junior high school levels. Interestingly, little research concerning measure learning has been conducted at the age levels where most instruction takes place.

It should be noted that we do not have considerable research evidence concerning the sequences of children's naturally acquired concepts in several different measure systems. For example, researchers in the Piagetian tradition have looked for horizontal décalages for important ideas like transitivity and conservation that generalize from one measure system to another. But the décalage research has not been oriented to instruction. It has not considered, generally, questions of how the learning of one measure system can be used to advantage in learning another measure system nor has it sought to identify the factors affecting this transfer. Rather, it has sought to determine if and when such learning of generalizations took place in a natural fashion without instructional intervention.

Mathematically, the similarities of measure systems are the most apparent characteristics of measure. But how have these similarities been taken advantage of in designing research on the learning of measure? The similarities are in terms of the structures of the measure systems. The similarities suggest the psychology of transfer.

We do not have refined notions about the nature of operant factors that may facilitate the transfer of structures. Dienes and Jeeves (1970) work examining finite groups and transfer is one of the first studies of structure and transfer. Although researchers (Branca & Kilpatrick, 1972; Brazier, 1974) have used the Dienes and Jeeves' approach to relationships between group structures, the evidence for what facilitates transfer between structures, if it happens, is quite inconclusive. Since the research is founded on the structures of finite groups, it reveals little information that is relevant to the structural transfer problem for measure systems. Brazier's study does, however, examine the role of student preference for a geometric or an algebraic mode of thinking in relation to the group problems; this factor may be of some significance in examining the learner's incorporation of number into measure systems.

The large majority of transfer studies have not been concerned with structures and structural properties. For example, psychologists have tended to focus on the learning of artificially simple concepts rather than examining the acquisition of complex relations and structures that are more typical of real measure systems. Thus, the researcher intending to use prior research as a base for design of new research problems is faced with using research concerned with the learning of finite groups or with the application of principles for the learning of relatively noncomplex concepts and ideas. Neither quite fits the learning of measure.

Given the large number of measure systems the child encounters and must use in matriculating through school, it appears that careful examination of the transfer process is potentially in the best interests of redesign of curriculum. This examination should be in terms of helping the student acquire new but similar structures efficiently.

The concept of structure has been identified as one of the primary characteristics of the post-Sputnik curricula in mathematics. The structural emphasis in the mathematical experiences of the child was intended to increase understanding and facilitate learning as well as bring mathematics to a current level of discourse and thinking styles. Interestingly, however, we have not evaluated the factors that might make structure a facilitator of learning. Perhaps the careful study of transfer of structures in a measure context will reveal a facilitating nature of structure in mathematical learning. Toward this end, the structures of various systems and their common elements have been identified (Blakers, 1967; Osborne, 1974). The next research task is to use what is known about transfer to investigate the effects of various instructional variables in terms of their ability to facilitate transfer between various measure systems. It is important to mention, however, that the complexity of measure systems ideas suggests that short term research efforts will not be as productive as longer term studies. Embedding the studies within a carefully conceived curriculum, such as Developing Mathematical Processes program, appears to improve the possibility of payoff in long term experiments.

A final category of needed research in the across transfer setting relates to the learning of more complex measure systems. Systems that are based upon composite functions, such as speed, have not generally been well researched. Examining transfer from less complex measure systems to such composite function systems or between composite function systems appears to offer some productive questions to pursue in a totally uncharted territory. The homomorphic-transfer approach also makes accessible different questions concerning how children cope with perceptually confusing systems of measure. For example, energy, mass, and temperature do not have visually perceptable characteristics, like length, area, volume, and angularity. But we have little insight into the perceptual factors involved. However, the across transfer approach provides a method of comparing the cognitive problems that learners may encounter.

The advantage of the homomorphic-transfer model is that it broadens the focus of measurement research. It provides a comprehensive logical analysis of the measurement process that incorporates a complete range of measurement operations. It identifies a number of significant variables that characterize different measurement operations. It provides a unifying theme for interpreting the results of various research studies, and it identifies a variety of basic measurement operations that have been ignored in measurement research. Furthermore, the homomorphic-transfer model encourages researchers to consider the complete measurement process--including the domain, range, and measure function.

One limitation of the homomorphic-transfer analysis of measurement is that it is based primarily on a logical analysis of the measurement process. Yet, Carpenter's review of measurement (in this volume) indicates that a strictly logical analysis of the measurement process does not account for many observed variations in performance and is not very useful in describing the effects of different training procedures. Therefore, in conjunction with the homomorphic-transfer analysis of measurement it is necessary to generate a more complete model of cognitive development.

Models of Cognitive Development

Flavell and Wohlwill (1969) propose that an analysis of cognitive development should incorporate a competence-performance distinction similar to Chomsky's model for language acquisition. The competence component of the model is the logical-mathematical structure of the domain, and the performance component represents the psychological processes by which the structures in the competence component get accessed and applied to specific tasks. The competence component is an idealized abstract representation of what is known or understood, whereas the performance component must account for the reality of stimulus variations, conflicting information, memory limitations, etc.

In Flavell and Wohlwill's model a child's performance for a given operation should be specified in terms of three parameters: P_a, the probability that the operation will be functional in a given child; P_b, the probability of the operation being applied to a given task; and k, the weight to be attached to P_b in a given child at a given age. The equation for the probability of a given child solving some particular task is:

$$P\ (+) = P_a \times P_b^{\ 1-k}$$

What is needed is research to identify the factors that influence P_b. It remains to be seen whether P_b can be quantified as a function of these factors. But whether they can be quantified or not, such factors as the constraints of the stimulus situation, the demands on memory, and the types of logical inferences required all affect performance; the effect of these factors should be included in any analysis of cognitive behavior.

One approach to an analysis of cognitive behavior involves the application of information processing techniques to describe and explain Piagetian operations (Baylor, Gascon, Lemoyne, & Pothier, 1973; Klahr & Wallace, 1970). Instead of analyzing behavior in terms of the logical and algebraic properties of the problem, the approach is to analyze the information processing requirements of the task. In other words, behavior is described in terms of the subroutines a child would need to apply in order to perform a given task. This procedure is somewhat analogous to analyzing the compilation and execution functions of a computer. This involves (a) encoding external stimuli, (b) assembly of task specific routines from a repertoire of fundamental processes, and (c) execution of the task specific routines. This not only forces the programmer to develop an explicit description of the behaviors involved in each task, but some sort of analysis on the demand of subroutines may provide an indication of the level of difficulty of each task.

This approach has generally been limited to situations like seriation, in which there is an abundance of observable action which can be taken to imply the application of certain strategies. It may be much more difficult to build information processing models for conservation, in which there is less overt action. Furthermore, there is also the difficulty that a child's logic is not always congruent with adult logic, and it may be very difficult to identify what aspect of the stimulus situation he is attending to and what specific heuristic strategies a child is really using. Nevertheless, many of the logical difficulties that occur in the development of measurement concepts seem to relate to information processing variables, especially if sufficient emphasis is given to the encoding process.

Although Baylor et al. (1973) do attempt to use the information processing analysis to construct a computer program to simulate children's behavior, the construction of artificial intelligence programs is not crucial for research in cognitive development. What is distinctive about their work is the analysis of behavior in terms of information processing requirements rather than strictly in terms of the logical and mathematical properties of the problem. The results of current research indicate that there are at least four major dimensions that should be included in such an analysis: (a) the constraints of the stimulus situation, (b) the demands on memory, (c) the types of logical inferences required, and (d) the types of responses required. These dimensions are comparable to the major components involved in the execution of a computer program. The stimulus dimension is analogous to the input component, the demands on memory can be analyzed in terms of use of short and long term memory, the type of logical inference required may be compared to the assembly and execution of the program, and the responses are comparable to the output of the computer.

There is a substantial body of research indicating that stimulus variables significantly affect performance on tasks testing basic measurement concepts (see the Carpenter paper in this collection). An especially significant stimulus variable seems to be the sequence of the cues and the presence or absence of conflict or perceptual support (Carpenter, 1975; Divers 1972; Jones, 1969; Trenary, 1972). The purpose of research in this area would be to identify what factors of a stimulus situation a child can attend to and to identify the effect of different stimulus variables at different stages of development. For example, in a study of conservation and measurement concepts (Carpenter, 1975) it was found that children can attend to numerical stimuli at least as readily as to perceptual stimuli. There was also some evidence that children in a transitional stage need the support of identifiable compensation in order to maintain conservation judgments.

A second factor that should be included in an analysis of cognitive behavior is the demand on memory. For instance, a study by Baylor et al. (1973) indicates that by equating tasks in terms of memory demands, the décalage between length and weight seriation disappears, and Roodin and Gruen (1970) found that the provision of memory aids significantly improves performance on a transitivity task.

The third and perhaps the most important dimension to be included in such an analysis is the type of logical inference required. Notions like centering, recognizing and resolving conflict, and simultaneously considering several variables are a few examples of types of logical inference that may be productive in describing cognitive behavior.

The stages of learning outlined by Inhelder (1972) to describe the results of a measurement training exercise are defined in terms of an information processing dimension similar to that described above. Subjects who showed no progress centered on a single dimension. If they were asked a length question or provided with length cues, they would center on length. If asked a number question, they would center on number and ignore length. There was successive application of two distinct systems, and these subjects saw no contradictions in their responses. In the second stage, both evaluation schemas seemed to be present simultaneously. Subjects were not satisfied with either a purely numerical or linear solution and turned from one solution to the next. However, they were not capable of a new solution that accounted for both dimensions simultaneously. Thus, although subjects were aware of the contradiction, they could not begin to resolve it. In the third stage, some attempt was made at integration, but the result was an inadequate "compromise solution." In the fourth stage, the different schemes were integrated and coherence was attained.

The fourth dimension is comparable to the output component of a computer. An analysis of cognitive behavior should include some consideration of the types of responses subjects are asked to make. Some children understand certain principles but are unable to verbalize them. Significantly earlier development of fundamental operations has been found with nonverbal assessment techniques compared to verbal techniques. Requiring that subjects provide valid justification for their responses significantly lowers the number of correct responses (Braine, 1959; Gruen, 1966; Sawada & Nelson, 1967).

Once some form of information processing model has been constructed, it should not only provide some insight into the relative difficulty of different tasks, but it should also provide a basis for analyzing the effect of training.

Training

Measurement learning studies have failed to identify the specific mechanisms of development. There are no critical experiments that confirm or preclude the major theoretical positions because the results of most training studies can be explained in either behaviorist or equilibration terms. One reason for this lack of success is that both theories are so general that they account for a wide range of behaviors. Therefore, while it does not appear likely that research will easily confirm or disprove major theories, no major theory appears to be readily verifiable. As a consequence, no current theoretical perspective provides a definitive structure for evaluating the potential effect of different systems of instruction.

Although contrary evidence exists, virtually every training procedure has in some sense been able to accelerate the acquisition of logical operations However, new research that does no more than demonstrate that training is possible will contribute little to our understanding of the development of measurement. At this point, the most promising objective for future research is to attempt to identify the specific effects of training. To date most training research has only demonstrated that certain procedures do or do not improve the overall performance of a group of children on a given set of tasks. Future research should include a more complete analysis of (a) the entering cognitive skills of each subject, (b) specific learning effects for subjects at different cognitive levels, and (c) the specific variables of the training situation that account for learning.

One study that dealt with all three of these factors is the length conservation training study reported by Inhelder (1972). All subjects were pretested for number and length conservation, and only those who passed the number conservation problem and failed the length conservation problem were included in the experiment. Due to the one-to-one clinical training procedures, the Genevans were able to specify cognitive levels of subjects in terms of their performance during the training as well as on the basis of the pretest on number and length. Therefore, they were able to describe the cognitive processes of the students for whom the training was successful and those for whom it was not and could give some explanation why some subjects learned to conserve as a result of the training and others did not.

A major goal of training research is to identify measures of cognitive development that can specify which training procedures are appropriate for children at given levels of cognitive development. The Genevans have concluded that learning is very unlikely for children who fail to conserve numerousness. Although this hypothesis needs further validation, some measure of subjects' ability to conserve number would seem to be a fundamental basis of classifying subjects.

Information processing capabilities would seem to be one of the best measures of cognitive development for determining the effect of training. Teach-test procedures like those employed by Montgomery (1973) represent one potential measure of information processing skills that are specifically related to the ability of subjects to profit from different types of instruction.

Measurement training studies should include comprehensive measures of the specific effects of training for subjects at different cognitive levels. Results should include measures of transfer and retention. Transfer tests should be carefully structured to include skills that develop at or before the stage the trained skill naturally develops. In other words, the results of training studies should provide some indication of whether training has resulted in a narrowly learned skill or whether subjects have actually achieved a more advanced stage of development. Furthermore, these results should be analyzed for subjects at each stage of development. Results should describe the cognitive levels of subjects who are successful and those who are unsuccessful in achieving different objectives of the training exercise.

Finally, research should attempt to identify specific variables of training that account for learning of subjects at different cognitive levels. The information processing requirements of different instruction is one possible basis of analysis. It might be hypothesized that children's ability to learn from a given training procedure depends on whether the information processing demands of the training procedure are appropriate for their level of development rather than whether they have learned the logical prerequisite concepts. Some support for this position is provided by Inhelder's study (1972). She found that subjects who were not successful in learning to conserve length were those who could not deal with the information processing demands of the training procedures, whereas presumably none of the subjects had acquired the prerequisite measurement operations.

Conflict is another potentially significant variable. Although it appears that active participation in a conflict creating situation is not necessary for learning to occur (Beilin, 1971), for the Genevans, conflict is a major factor in the development of fundamental operations. The effect of this variable should be investigated by systematically varying levels of conflict in training. For example, the instructional sequence outlined by Inhelder (1972) to teach conservation of length begins with the most complex task. The first task generally induced an incorrect response based strictly on length. The second task generally induced an incorrect response based strictly on number. The basis for the two responses were in direct conflict. It was not until the third problem that sufficient information was provided to resolve this conflict. Thus, this sequence seems designed to maximize conflict. Subjects were induced to begin with incorrect conflicting responses and subsequently attempted to resolve this conflict. If the sequence of the problems was reversed, the level of conflict would presumably be reduced. A comparison of these two instructional sequences might provide some insight into the effect of different levels of conflict on learning.

Reversibility is another variable worth investigating. Brainerd and Allen (1971) have proposed that the distinguishing characteristic of successful training studies is that reversibility is inherent in their training procedures and is absent in unsuccessful training studies. They maintain that even those studies that do not specifically train for reversibility actually demonstrate either overtly or covertly the inverse of specified operations. They cite as an example a study by Gelman (1969), in which they are able to identify inverses of operations used in training in different steps of the training procedures. Since reversibility was not intended to be the significant variable in this study, it would be a simple matter to redesign a parallel study that contained no reversibility. A comparison of the relative effect of these revised instructional procedures as compared with the original procedures would provide some test of Brainerd and Allen's hypothesis.

Many training studies have employed a single training procedure. Although they have been able to demonstrate that training is possible, most of these studies provide little insight into the relative effectiveness of specific training variables. Studies that have compared different types of training have often chosen such dissimilar procedures that comparison is very difficult. There is a need for more studies like those described above that provide systematic variation along a single dimension.

Measurement in the Mathematics Curriculum

The greatest deficiency in current research is the lack of studies relating the results of clinical studies to the instruction of measurement in the mathematics curriculum. Clinical studies have yet to provide a definitive picture of the development of measurement concepts or the effect of training in clincial settings, but it is doubtful that they ever will. However, there is sufficient clincial research to begin to study the implications of this research for the mathematics curriculum.

A fundamental question is the placement and sequence of topics. Research has identified a number of basic operations like conservation and transitivity that appear to be fundamental for the development of measurement concepts. But, it is not clear whether or not fitting instruction to coincide with the appropriate cognitive operations is the optimal method of instruction. The results of clinical studies indicate that instruction in measurement may be facilitated (rather than strictly depend on) the development of these operations. However, it is not clear how these results obtained in short, one-to-one clinical training sessions apply to the mathematics curriculum. At this point, the net result of extended instruction for preoperational children in fundamental measurement operations is not known. Research in school settings over extended periods of time is needed to identify the appropriate placement and sequence of topics for children at different stages of development.

A second basic question has to do with determining the effects of long term instruction on fundamental measurement operations. Training studies have generally been conducted over short periods of time. Consequently, the gains they have induced have generally been modest. A study by Gal'perin and Georgiev (1969) indicates that instruction over longer periods of time may yield substantial gains in the development of basic cognitive operations. Due to the extraordinary gains that this study claims to have achieved, careful replication is warranted.

A second approach is to evaluate the relative effectiveness of existing curricula like Developing Mathematical Processes or AAAS Science: a Process Approach that provide specific instruction in fundamental measurement processes. This evaluation should specifically focus on the development of fundamental operations like conservation and transitivity, and should test the generalizability of measurement learning.

Measurement studies conducted within regular school settings over extended periods of time require a substantial investment of resources, and it is difficult to maintain strict experimental controls. However, these studies are needed to obtain valid insight about what implications clinical research on measurement learning has for the mathematics curriculum.

Furthermore, research on basic measurement concepts should be a fundamental component of curriculum development. Isolated training studies that attempt to identify general principles of learning are always subject to the criticism that their results are only valid for the specific procedures employed. For example, there is no single set of training procedures that represents reversibility. Such studies conducted within the framework of curriculum development at least provide some indication of which of several alternatives should be pursued in the specific program.

Summary

Researchers in the area of teaching and learning of measure have a profound need for a model or a unifying conceptual basis that will accommodate to questions and problems concerning a variety of measure systems. The lack of such a model provides a fundamental characteristic of incoherence to much of the research related to the school-based learning of measure concepts and skills. Piagetian research has provided the beginnings of such a unifying conceptual system and many Genevan studies significantly contribute to our understanding of the learning processes of children. Unfortunately, however, the Piagetian analysis of measure ignores some of the salient features of the mathematical structures we call measure systems and, hence, has a limited potential for allowing researchers to find a comprehensive set of predictive factors affecting the learning of measure systems in general. The homomorphic nature of measure systems coupled with the psychological processes of transfer appears to possess a significant potential for revealing additional important factors in studying children's acquisition and understanding of measure systems. Further, it appears to provide a mechanism for studying cognitive processes encompassing what we already know about transfer and learning. Finally, it appears to be directly related to instructional questions and problems of significance to teachers.

It should be clearly understood that this does not imply that there is an overabundance of basic research concerning measurement concepts or that we are advocating abandoning directions of research that already have been established. Rather, we need to build upon this research and construct a more complete model of cognitive development. It is proposed that an analysis of behavior that takes into account the information processing demands of individual problems as well as their mathematical structure will be more productive than analysis that relies solely on logical-mathematical properties. At present, the most viable goal of such analysis is to generate a predictive model that can identify the sequence of acquisition of different tasks and can be used to evaluate the effect of instruction for children at different levels of development.

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On the Mathematical, Cognitive and Instructional Foundations of Rational Numbers

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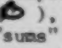
Perspectives on Rational Numbers

The sophisticated consideration of whole-part relationships is one of the significant intellectual achievements of mankind. Yet, fractions (and the rational numbers they represent) have been with us almost throughout recorded history. For example, in ancient Babylon (circa 2000 B.C.), written records indicated the existence of a sexagesimal precursor to our decimal-fraction. Nonetheless, it is evident that the Babylonian system was not a true place value system because what we know as rational numbers were expressed in a mixture of decimal and sexagesimal notation. What is today considered the whole number part of a decimal fraction was expressed in base 10, while the "fractional" part (<1) was shown in sexagesimal form (Wilder, 1968). This strange phenomenon arose from the adaptive absorption of one ancient Middle Eastern culture by another, and it has persisted even until today. For example, when decimals are used in angle measure--236 degrees, 27 minutes, 12.5 seconds--the result is obviously a decimal-sexagesimal expression of the fraction. Thus, we see that rational numbers and even features of their "arithmetic" have been used by civilizations for the past four thousand years. Consequently, it seems odd that instruction of children in this arithmetic should continue to be the object of current scholarly study.

The seeming contradiction of ancient "success" with rational numbers and modern consternation over instruction in rational numbers is perhaps most vividly evident in the Egyptian case. As seen in the ancient Ahmes papyrus, rational numbers were expressed in terms of unit fractions (Mainville, 1969). For example, 16/63 would be expressed as 1/7 [+] 1/9. Thus, 1/4 and 1/11 were expressed as and respectively. But, except for 2/3 (), rationals whose numerators were other than one were written as "sums" of unit fractions. Looking at this, a modern person is moved to say, "How strange that people would go to this level of complexity to express rational numbers!" One marvels even more at the tremendously complex applications of rational numbers as measures seen in the pyramids (Tompkins, 1971).

How is it that the ancients showed such intellectual facility and such engineering skill using such a "primitive" numeration scheme? How is it that our children labor over learning fractions to such an extent? It is clear that control over these mathematical ideas and the ability to use them to control and order reality is a result of a thorough understanding of the many interpretations of rational numbers. The facilities inherent in modern numeration are of no help if the student lacks the basic understanding of rational number ideas. Even with electronic calculators and computers, it is the mathematical understanding which gives students control and allows for applications.

In what follows, it will be argued that to understand the ideas of rational numbers, one must have adequate experience with their many interpretations. Most school curriculum materials simply treat rational numbers as objects of computation. Hence, children and adolescents miss many of the important interpretations of rational numbers. In particular, the "algebraic" aspects of the operations on rationals are lost. Yet, rational numbers present a face to face confrontation with algebraic problems because the child must:

(a) grapple with the notion of equivalence;

(b) cope with an operation "+" which in its algebraic form "works" the way it does mainly for axiomatic reasons and is no longer natural;

(c) work in a system where "+" and "x" are two distinct operations, abstractly defined (these two operations with rationals are analogous to "adding" lengths and composing functions); and

(d) work with the properties, particularly a general notion of inverse.

Consequently, rational number concepts are different from concepts associated with the study of natural numbers. Natural number concepts (and, to a certain extent, the operation of addition) arise out of the natural activity of children. And, multiplication is treated as a special form of counting or as repeated addition; so its "algebraic" nature is not apparent (although it has been observed historically that children have difficulty multiplying by 0 and 1--algebraic notions).

To adequately learn the algebraic aspects that are inherent in rational number concepts, the author will argue that a variety of experiences with diverse interpretations of rational numbers are necessary. But what are these diverse interpretations? While the following list is not exhaustive, it contains those interpretations which serve as a basis for the analysis of rational numbers given in this paper.

1. Rational numbers are fractions which can be compared, added, subtracted, etc.

2. Rational numbers are decimal fractions which form a natural extension (via our numeration system) to the whole numbers.

3. Rational numbers are equivalence classes of fractions. Thus, {1/2, 2/4, 3/6, . . .} and {2/3, 4/6, 6/9, . . .} are rational numbers.

4. Rational numbers are numbers of the form p/q, where p, q are integers and $q \neq 0$. In this form, rational numbers are "ratio" numbers.

5. Rational numbers are multiplicative operators (e.g., stretchers, shrinkers, etc.).

6. Rational numbers are elements of an infinite ordered quotient field. They are numbers of the form $x = p/q$ where x satisfies the equation $qx = p$.

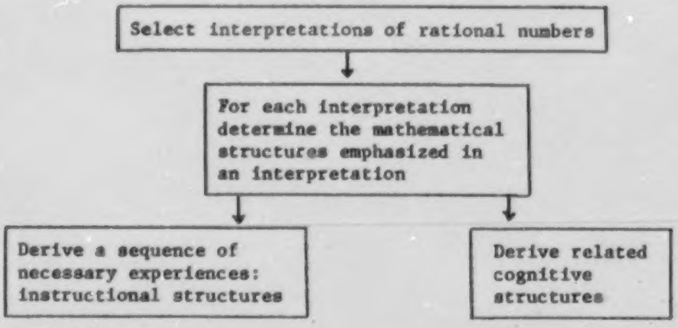
7. Rational numbers are measures or points on a number line.

Clearly these interpretations are not independent. Indeed, with appropriately defined operations and relations, they should be isomorphic. But each interpretation allows for the consideration of rationals from a different perspective. The major portion of this paper is devoted to the analysis of these interpretations.

If the goal is to understand how children and adolescents think about rationals or to understand necessary instructional moves for teaching rational numbers, what is the value of conducting a logical analysis of mathematical interpretations of rational numbers? It has been suggested earlier that, in working with rational numbers, children are dealing with mathematical structures which do not have an obvious basis in natural thought. Hence, a study of the natural thought of a child would not be adequate for consideration of the development of rational number ideas.

Sorting out some of the most important logical interpretations of rational numbers should contribute to future research in a variety of ways. First, only a limited amount of research has been conducted concerning any of the various interpretations of rational numbers (e.g., Piaget, Inhelder, & Szeminska, 1960; Steffe & Parr, 1968), and some interpretations have been almost totally neglected. Second, by failing to take into consideration the unique salient features of each interpretation, teachers and researchers have encountered difficulties that could have been logically anticipated (e.g., instruction based on the "multiplicative operator" interpretation of rational numbers does not furnish a good basis for understanding addition of fractions). Third, by investigating similarities and differences between various rational number interpretations, it should be easier to form generalizations from the results of studies that focus on individual interpretations. Fourth, the logical analysis should suggest several alternative sequences of experiences which might contribute to the acquisition of various rational number interpretations. It is hoped that future curriculum development will take into account the various interpretations and the related instructional sequences that are suggested, and that under these conditions a more productive study of rational numbers will be a reality for children.

The strategy of this paper is summarized below.


Select interpretations of rational numbers
For each interpretation determine the mathematical structures emphasized in an interpretation
Derive a sequence of necessary experiences: instructional structures
Derive related cognitive structures

Rational Numbers as Fractions

Fish (1874) in the preface to his book, The Complete Arithmetic, advises teachers that calculation as a faculty to be developed must precede reasoning. Although, later, logic and reasoning were to be encouraged, perfect accuracy in computation was to be maintained.

In many respects, Fish's admonitions relate well to his approach to rational numbers. Little or no attention was paid to systematic aspects. Fractions were objects of calculation. Attention to symbolic manipulation was the concern of instruction, as was the memorization of numerous terms. Even the applications presented by Fish (1874) seemed contrived.

> In 108/9 acre, how many acres? (p. 106)
>
> A man paid $25 7/8 for a watch and sold it for $6 1/4 more than he gave for it. What did he sell it for? (p. 110)
>
> A farm is divided into 4 fields; the first contains 29 7/12 acres, the second, 50 16/21 acres, the third, 41 6/7 acres, and the fourth, 69 3/4 acres. How many acres in the farm? (p. 111)

It is clear that these were not problem solving exercises, but simply computational exercises. How were these exercises to be done? Solutions were to be deduced from memorized principles and rules: "Fractions can be added only when they have a common denominator" (Fish, 1974, p. 110)

The topics and concerns of modern text treatments of fractions are frequently similar to those of their predecessors of 100 or more years. Gone are many of the "absurd" problems. Gone is the emphasis on developing the "faculty" of calculation. In their place is some emphasis on intuition derived mainly from an increased use of visuals.

This intuition, along with an appeal to principles of "modern mathematics" (e.g., equivalent sets of fractions, etc.), form the basis of concept development. Yet, in Ebos, Robinson, and Pogue (1975) we find statements such as, "To add fractions with unlike denominators . . .we have another step to do. We must find equivalent fractions which have the same denominator"(p. 38). This quote is suggestive of the basic similarity which this text has with its predecessor of 100 years. One notes that regardless of the different approaches, the main objectives of the two texts are computational and definitional. In both cases addition, subtraction, etc., are seen as procedures and not as operations in an algebraic sense. There is little interest in the mathematical nature of rational numbers or in the system of rational numbers. The experiences suggested are aimed at procedural skill and not thought of as a basis for later work in algebra or analysis.

When the computation with fractions is the focus of instruction, the learner is faced with a sequence of skills to learn. In this sequence there are frequent ordered subsequences leading hierarchically to a particular skill. For example, in Fish (1874) we find this sequence of topics leading to addition of fractions:

(a) reducing fractions to higher and lower terms (in modern parlance--equivalent fractions);

(b) reducing integers or mixed numbers to improper fractions;

(c) reducing fractions to equivalent fractions having a common denominator;

(d) finding least common denominators of fractions; and

(e) adding fractions.

This sequence was further interspersed with principles and operating rules for handling all cases of particular processes. So, with new terminology accompanied with more intuitive instructional procedures, this list closely parallels hierarchies of behavioral objectives in many modern "learning packages" or "uni-paks." Thus, the mathematical structure most evident when rationals are interpreted as fractions is a sequence of procedures or algorithms.

With this perspective on rational numbers, the necessary cognitive structures and the closely related instructional procedures focus on carefully developed skills with algorithms. As with other "skills," when a learner fails to exhibit appropriate behavior, the common response is diagnosis via task analysis. Although this process is instructionally fashionable today, it reached its height in the 1930's and 40's in the work of persons such as Brueckner and Grossnickle (1947). While not resorting to elaborate sets of associations à la Thorndike, Brueckner and Grossnickle, nonetheless, laid out a very detailed analysis of skills with fractions. There were four major types of addition problems:

(a) like denominators;

(b) unlike but related denominators (1/2 + 3/4);

(c) unlike and unrelated denominators with no common factor (1/7 + 1/11); and

(d) unlike and unrelated denominators with common factors.

For each type there were five kinds of fractions involved:

(a) proper reducible;

(b) proper nonreducible;

(c) improper fractions as mixed numbers with reducible fractions (1 9/6 = 2 1/2);

(d) mixed numbers with nonreducible fractions (8 5/4 = 9 1/4); and

(e) mixed numbers with fractions changeable to a whole number (p. 311).

Each of these settings was further analyzed into eight cases according to the sum obtained. For example:

(a) two fractions having a sum less than 1;

(b) two fractions having a sum greater than 1;

(c) mixed numbers with a fractional sum less than 1; and

(d) fractions whose sum is 1.

Thus, there were 4 x 5 x 8 = 160 different addition types. The authors, Brueckner and Grossnickle, of course did not envision these skills as independent.

> When the pupil has learned to find each kind of sum (a-e) in adding like fractions (type 1), he has learned the general procedure to use for each of the other 3 types, after unlike fractions have been changed to a common denominator (p. 311).

As noted previously, the above type analysis leads one to diagnose errors made by children in hopes of specifying a reteaching task. However, this search is often futile because emphasizing computation does not necessarily lead to instructional procedures relating to the nature of rationals or the operations. This is seen in a study (Gardner, 1941) which used Brueckner's error analysis procedure to study some 24,000 computations with fractions done by Scottish children. In the four processes, the most common errors were considered to be due to lack of comprehension of the process or use of the wrong process. These accounted for 36 percent of the errors in addition and over 50 percent in division.

To summarize, the mathematical structure paramount under the interpretation of rational numbers as fractions is a highly specific set of procedures or algorithms. These algorithms focus on the manipulation of fractions at the symbolic level. The mathematical goals of this interpretation can be thought to be self-contained. That is, these objectives need not be thought of as prerequisite to later mathematics, although they certainly can be looked at as a basis for manipulation of algebraic expressions.

The corresponding cognitive structure is a set of skills. It is not necessary under this interpretation to assume any other structures underlying the skills. The prerequisites for these skills would be skill in computation with whole numbers and not developed concepts of part-whole relationships or proportionality.

The major instructional strategy is diagnosis and remediation both based on elaborate task analysis.

Rational Numbers as Equivalence Classes of Fractions

The previous section of this paper described the arithmetic of fractions as an interpretation of rational numbers. This section will focus on the rational number as a set of ordered pairs of integers. The first step in this development is to define equivalence of ordered pairs (fractional form): $a/b \approx c/d \Leftrightarrow ad = bc$. In other words, equivalence of ordered pairs is defined in terms of equality of integers (or whole numbers). Using equivalence, rational numbers are defined as per Scandura (1971),

one-half: $\{1/2, 2/4, 3/6, \ldots\}$ or

one-half: $\{a/b \mid a/b \approx 1/2\}$.

Then the set of rational numbers is the set of equivalence classes of fractions a/b, $b \neq 0$, under the relation $\approx$. One can picture such a set of pairs as in Figure 1.

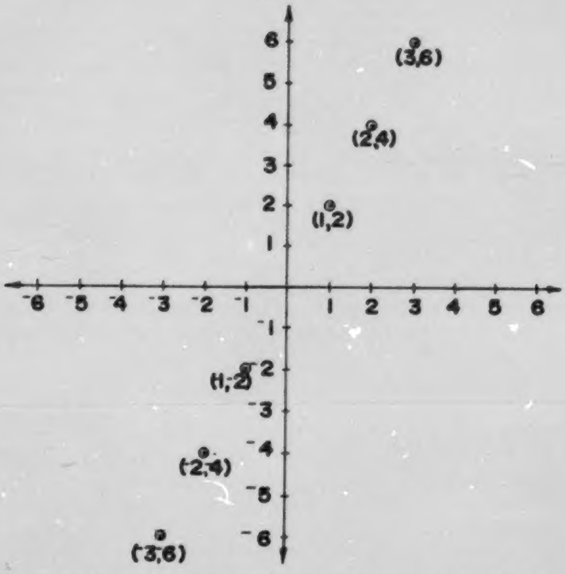


Figure 1. Several members of the equivalence class for the rational number 1/2.

As with any set of mathematical objects, the interesting thing to do is to work with them. For example, if addition is defined as follows:

$$a/b + c/d = \frac{ad + cb}{bd}$$

where a/b, c/d are representative elements from equivalence classes, there is a need to show that $3/6 + 4/9 = 51/54$ and $1/2 + 4/9 = 17/18$ are equivalent. That is, it must be shown that addition (or multiplication) is quite unique.

It should be noted that it is the definition of operations which differentiates the rational numbers (a field) from the set of ratios or rate-pairs (which behave somewhat like elements of a vector space). This distinction is reconsidered later in the paper.

The final activity in this interpretation is the study of the properties of rationals under operations. This study makes extensive use of the equivalence class notion in establishing the properties of an ordered field.

In coping with the mathematical notions of the equivalence class interpretation, the child must develop certain underlying concepts. One principle concept which must be developed is the notion of an ordered pair of numbers to represent reality. This notion entails three phases. The child must learn to observe reality in terms of coordinates.

That is, he must observe a situation, see its parts, and identify these parts in order. The second phase is a representation phase; the child must learn appropriate symbolic structures with which to represent a coordinated reality. A closely related third phase involves learning to correctly identify the symbolic representation with the ordered parts of reality. In particular, with rational numbers the child must learn to identify part-whole situations, learn verbal and numerical codes for these, and learn to correctly identify a code (fraction) with a part-whole setting. As a cognitive capstone of this ordered pair concept set, the child must realize that a part-whole setting can be seen in a set of equivalent ways, and that the various fractions which represent the elements of this set are equivalent.

The basic instructional strategy related to this conceptual development is experience with a wide variety of part-whole settings. Scandura (1971) suggests four such settings. The first of these (and, for Scandura, the most important one) is the static comparison between a set and a subset thereof--which he identifies as a state-state comparison.

{A A A B B B}   {(AAA) (BBB)}

The picture above depicts two state-state comparisons. Looking at the left-hand set, one sees that in this set of six elements, three are A's. In the right-hand picture, the same set is seen to have two equipotent subsets. Thus, of two sets, one is a set of A's. From such experiences, the notions of fractional representation (3/6, 1/2) and equivalence (3/6 = 1/2) develop.

While the above setting represents observations of static phenomena, the other three settings entail the observation of the active manipulation of parts and are briefly summarized below.

Type	Action	Symb. Rep.
State-Operator	Divide 3 cookies among 5 persons	3/5
Operator-State	Use 5 of a dozen eggs	5/12
Operator-Operator	Cut a pie in eighths, serve 5	5/8

Instructional settings based upon these part-whole comparisons form the basis for understanding of the ordered pair notion including the idea of equivalence.

The notion of ordered pair representation of fractional situations culminates in the understanding that these ordered pairs are numbers. This understanding calls for at least two things. First, the child must be able to relate this new set of numbers (a set of equivalence classes of ordered pairs) to the whole numbers with which he has dealt: In what ways are the new numbers different, the same, etc.? Does this new set contain the old (e.g., 4 ≈ 8/2)?

The second ability or concept set which a child must possess to treat ordered pairs as elements of a number system is the notion of operations consistent with the fractional and equivalence notions discussed above. Within the context of ordered pairs, Scandura (1971) sees the operations are derivatives of these notions as is illustrated in Figure 2a and 2b.

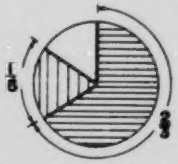


Figure 2a. 2/3 + 1/5

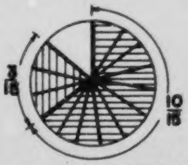


Figure 2b. 10/15 + 3/15

In Figure 2a, we see a part-whole diagram which can be represented by 2/3 + 1/5. Figure 2b shows the use of equivalence in dividing the thirds into five pieces and the fifth into three pieces. This can now be written 10/15 + 3/15 = 13/15.

To master the subdivision of parts concept that is inherent in developing concepts of operations (i.e., addition, multiplication), the ability to partition is needed. This notion is illustrated in the following activities:

1. Here are 15 plants and 5 pots. If all are the same, how many plants per pot?

2. Divide this rope into 5 equal pieces.

3. Divide these crackers among 4 people.

It should be noted that a general notion of partitioning includes both discrete and continuous quantities.

When interpreting rational numbers as ordered pairs, the principle mathematical idea is that of equivalence class. From this mathematical idea flows the notion of operations on rationals and also the properties.

Working on rationals within this interpretation, the child must be able to assign a pair of numbers to a part-whole situation. This, of course, entails the ability to logically handle the part-whole relationship in both the discrete and continuous cases. The ability to handle class inclusion may be very important in the former case, while partitioning plays a role in the latter. Partitioning also plays a role in the ability to see a part-whole situation in equivalent ways which forms the basis for the notion of equivalence. With respect to the notion of equivalence within the ordered pair interpretation, the ratio aspect comes only as a formal capstone. The concepts of operations are derived from the notions of ordered-pair and equivalence with partitioning again being a principle ability.

Rational Numbers as Ratio Numbers

A second interpretation of rationals which leans heavily on the ordered pair notion can be developed in response to the following question.

> In a comparison in which 1 is paired with 8, 2 with 16, 3 with 24, etc., what is paired with 1?

Put more symbolically, $(x, 1) \approx (1, 8) \approx (2, 16) \approx (3, 24)$. Skemp (1964) uses this question as the basis for developing another interpretation of rational numbers.

In response to this query, x can be thought of as the number indicated by the pair (1, 8) or to avoid confusion 1/8. In considering the proportion $(y, 2) \approx (1, 8)$, y should be twice x, or

$$y = 2 \cdot 1/8.$$

If this is interpreted as repeated addition, we get $y = 1/8 + 1/8$. The classic question now arises. Is $1/8 + 1/8 = 2/16$? Using knowledge of proportions and in particular equivalence, we get

$$(y, 2) \approx (1, 8) <=> 8y = 2$$

or $y = 2/8$. Thus, $1/8 + 1/8$ should equal $2/8$ to be consistent with this definition of equivalence.

Operations on the ratio numbers proceed from the notion of equivalence and equivalence classes. For example, we know the elements equivalent to (2, 3) can be generated by mutliplying 2 and 3 by any constant k. This class is

$$\{(2, 3), (2x2, 2x3), (3x2, 3x3), (8, 12), (10, 15), \ldots\}.$$

Similarly, the class equivalent to (1, 5) is

$$\{(1, 5), (2x1, 2x5), (3, 15), \ldots\}.$$

The notion of equivalence also provides a rule for filling in the following number line:

x y z ...	1 ...	2 ...
1 2 3 4 5 ...	15 ...	30 ...

For example:

(x, 1) = (1, 15) so x = (1, 15) or 1/15

(y, 2) = (1, 15) so y = (2, 15) or 2/15

(z, 3) = (1, 15) so z = (3, 15) or 3/15.

The correspondence becomes

1/15	2/15	3/15	4/15 ...	10/15	11/15	12/15	13/15 ...	1
1	2	3	4	10	11	12	13	15

Addition can be defined using this number line. For example, to add 2/3 + 1/5, we look at the equivalence classes for 2/3 and 1/5. In them we find 10/15 and 3/15. Looking at our double labeled number line, we see that 10/15 + 3/15 would correspond to 10 + 3. Thus, the sum is 13/15. Combining the ideas from the last two paragraphs gives a strategy for addition: Namely, study the equivalence classes and add pairs with the same second element. One can see the arbitrary nature of the addition strategy creep into the above approach. On the basis of the ratio notion, the much more natural definition is the vectoral one:

$$(a, b) + (c, d) = (a + c, b + d).$$

As in the previous interpretation, operations in the ratio interpretation are developed on the basis of a relationship to counting numbers.

One can see that, from the point of view of the child, ratio numbers are a sophisticated entity. The basis for development is the notion of ratio based upon symbolic control of the proportionality schema. As Lovell (1971b) and others have shown, this schema or capability is not fully developed until later adolescence. The operations on rationals as developed under this interpretation, although algorithmically simple, are sophisticated in concept. Ability with them demands of the child the ability to handle equivalence symbolically and to "transfer" number line concepts to these ratio numbers. The child must be able to "scale" a number line in any number of ways.

The instructional structures which will promote these notions are primarily symbolic. Dividing the unit intervals in a number of different ways would help establish the positional notion of the ratio number as well as illustrating equivalence. For example, folding a one meter strip of adding machine tape into three and six parts would lead to the positional idea of 2/3 and the equivalence of 2/3 and 4/6. In this respect, the ratio number interpretation is a sophisticated version of the measurement interpretation which will be considered later in this paper.

Rational Numbers as Operators or Mappings

In the previous two parts of this paper, rational numbers were associated with the mathematical notions of computation, sets, and ratios. Another mathematical interpretation of rational numbers is based on mapping or operators. This can be seen either as discrete mapping of finite sets onto finite sets, or it can be seen in the mapping of the Euclidean plane onto itself. The latter is illustrated in Figure 3.

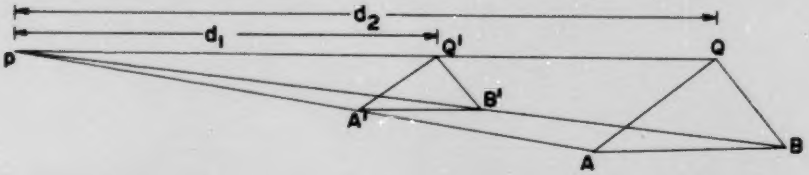


Figure 3. Rational numbers as operators.

Under this mapping, the point Q in the plane is mapped onto point Q' collinear with P such that $PQ'/PQ = k$. In Figure 3, $k = d_1/d_2 = 2/3$. Thus, the rational number 2/3 becomes associated with a mapping which transforms line segments into line segments 2/3 their original length. This mapping of the plane onto itself is called a dilatation or a similarity map and is determined by the point P and the ratio of similitude, in this case 2/3.

According to the mapping interpretation, one can think of the rational number p/q as a transformer or operator changing a geometric figure into a figure p/q times as big. The relationship between the object and its image depends on p/q. If $|p/q| < 1$, the image is smaller than the object. If $|p/q| > 1$, the image is larger. If $|p/q| = 1$, the image and object are congruent.

Under the mapping illustrated in Figure 3, objects six units from P would be mapped onto objects four units from P. Thus, the 4/6 operator is the same as the 2/3 operator, or $4/6 \approx 2/3$. Indeed, there are an infinite number of operators equivalent to 2/3. Another concrete way of considering equivalent rational fractions with respect to dilatations is to compare the length of a segment with its image segment. Thus, under a "2/3" transformation, a segment of length twelve would map onto one of length eight, a segment of length thirty-six with one of twenty-four, and so on. The fractions representing these comparisons, 8/12, 24/36, etc., would be equivalent and would be of the class "2/3."

The fact that there are an infinite number of operators equivalent to 2/3 is more vividly seen in the "finite set" representation (Dienes, 1971). Suppose two boxes of crayons are given to every three children. This illustrates, in Dienes' parlance, a two-for-three operator. Since under this condition, twelve children would be equipped with eight boxes, we could say that these sets of twelve and eight are in the fractional state 2/3. There is an infinite set of sets in this state. Equivalence can also be seen in operators. If we have a "four-for-six" operator and twelve children, we again need eight boxes. Thus, twelve and eight are also in the 4/6 state. Thus, the operators two-for-three and four-for-six, doing the same thing, are equivalent.

What happens when one such rational operator is followed by another? For example, take a segment twenty-four units long. Under a 2/3 operator or mapping, this is paired with a segment sixteen units long. If this new segment undergoes a 1/4 mapping, the new image is four units long. Looking at the twenty-four unit state and the final four unit state, one might say that the composition of 2/3 followed by 1/4 leads to a 4/24 mapping, or equivalently 2/12 or 1/6 mappings.

If two mappings, 2/7 and 3/5, are combined, the relationship to the common notion of multiplying fractions is obvious.

$$35 \text{ units} \longrightarrow \textcircled{2/7} \longrightarrow 10 \text{ units} \longrightarrow \textcircled{3/5} \longrightarrow 6 \text{ units}$$

$$35 \text{ units} \longrightarrow \textcircled{6/35} \longrightarrow 6 \text{ units}$$

$$14 \text{ units} \longrightarrow \textcircled{2/7} \longrightarrow 4 \text{ units} \longrightarrow \textcircled{3/5} \longrightarrow 12/5 \text{ units}$$

$$14 \text{ units} \longrightarrow \textcircled{6/35} \longrightarrow 12/5 \text{ units}$$

This orientation leads naturally to an exploration of a/b followed by c/d or in expression form: a/b ⓕ c/d. For example, a/b ⓕ b/a = ? p/q ⓕ 1/1 = ? These questions lead to the notions of identity and inverse operators. Similarly, the use of three fractional operators and order of operators leads to the associative and commutative properties. Dienes (1971) points out that these properties transfer naturally to the related fractional states. Thus, the notion of fraction as an operator leads to the idea that the rational numbers form a group under multiplication.

One problematic aspect of focusing on the operator interpretation of rational numbers is the attempt to explain "addition" in these terms. This problem arises because this model of rationals is again intimately tied to the ratio notion, and the operator is essentially multiplicative. As Dienes (1971) suggests "the multiplication procedure is much simpler, from the structural point of view, than the addition procedure" (p. 152).

The "trick" of addition is in choosing an initial state such that operations by two fractional operators yield whole number results. Thus, if we are looking at 1/3 and 2/5, an initial state of 30 would be satisfactory. Under 1/3 we get: $30 \longrightarrow \textcircled{1/3} \longrightarrow 10$. Under 2/5 we get: $30 \longrightarrow \textcircled{2/5} \longrightarrow 12$. Adding the final states we get: $30 \longrightarrow \textcircled{1/3 + 2/5} \longrightarrow 22$. Now it is easy to see that 30 is mapped to 22 as follows: $30 \longrightarrow \boxed{11/15} \longrightarrow 22$. Thus, $1/3 + 2/5 = 11/15$. Again, to quote Dienes (1971) "It is in the search for this initial state that the technique of finding the common denominator is disguised" (p. 152).

One concept which an operator model of rationals illuminates is division. Thus, to divide 2/3 by 7/8, the question is "What operator k will take 7/8 to 2/3?" We know from previous work that: $1 \longrightarrow \boxed{2/3} \longrightarrow 2/3$, and $7/8 \longrightarrow \boxed{8/7} \longrightarrow 1$. Combining, we get: $7/8 \longrightarrow \boxed{8/7 \times 2/3} \longrightarrow 2/3$. Thus, $2/3 \div 7/8 = 2/3 \times 8/7$, or in general, $a/b \div p/q = a/b \times q/p$.

There are many cognitive structures which a child must develop in working in this interpretation of rationals. As in the last two interpretations, the child must associate a pair of numbers as an entity, but since this develops rather naturally in the activities noted, it will not be considered further. Three notions are critical to this interpretation. The first is the notion of proportion. As has been suggested by Copeland (1974) and Lovell (1971b), the schema of proportionality is not fully developed until the stage of formal operations. However, the rational number notions in this interpretation can be developed as concrete generalizations about a large number of concrete situations. Thus, these notions from the point of view of the child can be considered preproportional. It should also be noted that the fraction notion in this interpretation is based on the quantitative comparison of two sets or two objects; hence, part-whole or class inclusion notions are not central to the interpretation.

The second structure which a child must develop is that of composition. Thus, the child must see one transformation followed by another as a whole. Further, he must be able to conceptually replace these transformations by a third (their product). Because the operation of addition is not a composition of this form (that is, a composition of functions), Dienes considers it structurally more difficult than multiplication. Indeed, within this interpretation, this is true from the point of view of the child.

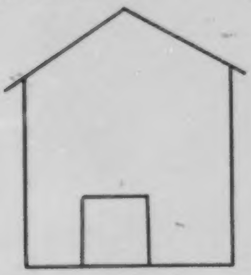
The third structure which is central to the operator notion is that of properties, particularly those of identity and inverse. Underlying this understanding would be a generalized reversibility notion, that is, the ability to give reversibility arguments in many different identity situations.

Without further elaborating on the mathematics generated by focusing on rational numbers as operators or on the related cognitive structures, let us consider the instructional underpinnings. If one considers the continuous model (i.e., dilatations of the plane), the obvious related instructional activity is work with similar figures. Perhaps the simplest form that this activity can take is the making and measuring of scale drawings on graph paper. Children from the ages of eight years or so bring numerous modeling experiences to mathematics class; hence, these activities fall within the natural frame of reference of the child.

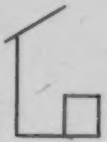
This kind of mapping can be simplified by simply pairing segments using fraction operators. These activities or "realities" for the child are illustrated below.

Story 1

Here is a house. It belongs to Mr. Jones.



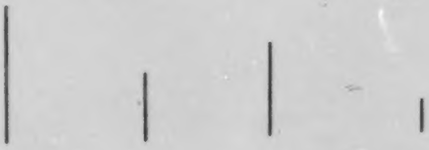
Here is part of another house. It belongs to Mr. Smith and is smaller than Mr. Jones' house, but is exactly like it. Can you finish it?



How tall is Mr. Jones' house?________
How tall is Mr. Smith's house? ________
Mr. Jones' house is ________ times as tall as Mr. Smith's house.
The side wall of Mr. Smith's house is 5 meters. How tall is the side wall of Mr. Jones' house? ________ meters.

Story 2

These are lines (pardon the mathematics) in Line Town.



Bill Line is 4 cm tall. Jill Line is twice as tall. How tall is Jill? ________. Bill Line is twice as tall as Tiny Line. How tall is Tiny? ________. Giant Line is 16 cm tall. He is twice as tall as ________ Line.

Two things should be noted concerning programs of activities as suggested above. While Story 2 is simpler than Story 1, in some ways, the one dimensional situation is more "abstract" in the sense of being outside the range of experience of the child.

This "abstract" nature is related to the second point. Story 2 demands a more direct knowing of the notion of proportion than does Story 1. Particularly when dealing with equivalence, proportion is central to approaching rationals through a (multiplicative) operator approach. The activities above must be considered "preproportional" while a study of the rational numbers as abstract members of a system must surely be "postproportional."

Dienes (1971) uses finite sets and exchange games as a means of making fractional operators concrete. Using the Dienes notion, there are numerous distributing problems which can be made up involving exchange, just as correspondence can be seen in "giving one to each" activities. Exchange activities, such as two balls are given to each of four players in tennis (2 for 4 operator) or seven magazines for three students in art class (7 for 3 operator) are easy to generate. Similarly, equivalence and the properties of multiplication are easily simulated. It should not be considered that such games are the whole story. Once again, these games are prerational number games. However, these games are in important ways psychologically simpler than the stories above, because the notion of proportionality arises in the simpler form of measurement division. That is, the child simply has to divide the balls into sets of two and the players into sets of four.

Again, it can be asked, "What are activities which precede those noted above?" Dienes' answer would involve games like the following:

1. There are 18 students in the room. How many teams are there for art class if 3 are on each?

2. The tennis club has 20 members. It takes 4 persons to play doubles. How many doubles can there be at one time?

3. Here are 20 animals. How many cages are needed if 5 animals reside in a cage?

4. Here are 15 cars at the landing. If the ferry can take 3 per trip, how many trips must it make? (Nelson & Sawada, 1975)

5. How many ________ rods make up a ________rod?

These activities could all be classified as measurement division activities. Measurement division activities require only the ability to make sets of a required number--that is, some form of counting algorithm.

In summary, we can ask the question, "Where does focusing on the 'operator' or 'mapping' interpretation of rational numbers lead?" We have seen that this interpretation leads nicely to the notion of multiplication of rationals and that it leads naturally to the group properties. It does not naturally lead to considering rational numbers as measures or the related additive activities, and because of its ratio basis, it does not naturally lead to the field axioms. Thus, the primary contribution of the operator notion is an algebraic one.

There are three primary cognitive structures associated with the operator or mapping interpretation of rationals. The first of these is the ability to compose--that is, to conceive of the product of two operations as a whole, representable by some new operation. The second is a general notion of reversibility which can support the abstract notions of inverse and identity. The third is proportionality. However, within the context of a discrete model of this interpretation, preproportionality notions serve to support the concepts of rational numbers.

Rational Numbers as Elements of a Quotient Field

In the previous section, rational numbers were interpreted as operators or mappings which led to consideration of their algebraic nature. In this section, the field axioms will be assumed, and rational numbers will be interpreted as elements of a quotient field.

Elements of a quotient field are numbers of the form b/a which represent solutions to equations of the form $ax = b$, where a and b are integers. Following Birkhoff and MacLane (1953), it can be established that in a field such quotients are possible and unique ($a \neq 0$).

The following theorems can be established for quotient fields.

1. $m/n = p/q \Leftrightarrow mq = np$
2. $(m/n) \pm (p/q) = \dfrac{mq \pm np}{nq}$
3. $(m/n) \times (p/q) = mp/nq$
4. $m/n + (-m/n) = 0$
5. $(m/n) \times (n/m) = 1$

The establishment of these theorems follows from simple work with equations. For example, relating to theorem 1:

$$\begin{aligned}
 & 2/3 = 4/6 \\
\Leftrightarrow\quad & 2/3 \times 3 = 4/6 \times 3 \\
\Leftrightarrow\quad & (2/3 \times 3) \times 6 = (4/6 \times 3) \times 6 \\
\Leftrightarrow\quad & 2 \times 6 = 4/6 \times (3 \times 6) \\
\Leftrightarrow\quad & 2 \times 6 = (4/6 \times 6) \times 3 \\
\Leftrightarrow\quad & 2 \times 6 = 4 \times 3
\end{aligned}$$

Thus, using field properties with equations gives a means of testing equality of quotients.

Similar arguments can be used to generate a general algorithm for addition. For example, to interpret the algorithmic meaning of addition, consider the following:

If $x = 3/4$ and $y = 2/7$, what is $x + y$?

$$\begin{array}{rlcrl} & x = 3/4 & \qquad & & y = 2/7 \\ \Leftrightarrow & 4x = 3 & & \Leftrightarrow & 7y = 2 \\ \Leftrightarrow & 28x = 21 & & \Leftrightarrow & 28y = 8 \end{array}$$

Combining we get:

$$28x + 28y = 29$$

$$28\ (x + y) = 29$$

$$x + y = 29/28$$

$$\therefore\ 3/4 + 2/7 = 29/28$$

In general, an addition algorithm for m/n and p/q is generated as follows:

$$\begin{array}{rlcrl} & x = m/n & \qquad & & y = p/q \\ \Leftrightarrow & nx = m & & \Leftrightarrow & qy = p \\ \Leftrightarrow & qnx = qm & & \Leftrightarrow & qny = np \end{array}$$

Combining we get:

$$qn\ (x + y) = qm + np$$

$$x + y = \frac{qm + np}{qn}$$

$$\therefore m/n + p/q = \frac{qm + np}{nq}$$

Similarly, if m/n and p/q are quotients, it is a simple application of equation solution ideas to generate multiplication:

$$\begin{array}{rlcrl} & x = m/n & \qquad & & y = p/q \\ \Leftrightarrow & nx = m & & \Leftrightarrow & qy = p \end{array}$$

Combining we get:

$$nxqy = mp$$

$$\Leftrightarrow \quad nqxy = mp$$

$$\Leftrightarrow \quad xy = \frac{mp}{nq}$$

$$\therefore m/n \times p/q = \frac{mp}{nq}$$

Thus, if rationals are defined to be the set of pairs of integers b/a which satisfy equations of the form $ax = b$ for any a and b, $a \neq 0$, then rational numbers form a quotient field and equivalence, addition, multiplication, and their properties *must be defined as suggested above*.

The above field of rationals is ordered by defining $a/b > 0$ to mean that the integer $a \cdot b > 0$ (Birkhoff & MacLane, 1953, p. 49). This arises naturally as follows: $3/5 \times 5^2 = 3 \times 5$. Since 5^2 is positive, 3/5 and 3×5 must have the same sign--in this case both positive. Similarly, $3/(-5) \times (-5)^2 = 3 \times (-5)$. Since $(-5)^2$ is positive, $3/(-5)$ and $3 \times (-5)$ must have the same sign--in this case, negative.

The quotient field interpretation of rational numbers clearly relates rationals to abstract algebraic systems. On the surface, however this interpretation would seem the most remote from school mathematics. Yet Freudenthal (1973), in *Mathematics as an Educational Task*, argues that the quotient field interpretation is the most meaningful setting for rational number study. Discussing operations on rational numbers he argues as follows:

> The method follows a clear pattern, and at the same time they [the operations] are meaningful processes. There is not any question whether numerators or denominators are equalized, whether fractions are inverted or not. ...One may regret that the intuitivity of fractions has been lost *though it is* doubtful whether it ever existed. (pp. 226-227, emphasis mine)

Freudenthal (1973) further argues that this approach defers the necessity of defining rational numbers as equivalence classes "the didactic value of which is problematic" (p. 227).

As a further argument in favor of this approach, Freudenthal compares rationals with rational expressions. This approach outlined above, extending the integral domain of the field of quotients (integers → quotients of integers) parallels that of extending the polynomials to rational expressions. Thus, the development of the rationals is a precursor to algebra or, indeed, marks the first study of algebra. This notion is a recurring theme that will be returned to later in this paper.

It is clear that this interpretation of rational numbers is not closely related to the natural thought of the child. As suggested above, the operations on and properties of rational numbers are developed in a deductive manner. Thus, the adolescent must be capable of formal thinking or reasoning from a hypothesis in order to appreciate this particular rational number interpretation. In particular, he must be confident that equations behave consistently and be capable of generating and working with implications in a symbolic form.

As a precursor to this more formal deductive ability, the child must be capable of what might be called predeductive thinking, or concrete deduction. Given the restrictions or "assumptions" of a situation, the child must be able to draw conclusions or see patterns implied by it.

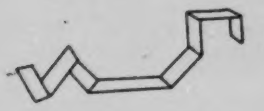
The major cognitive structure underlying the notion of quotient is partitioning. To reiterate what has been said earlier, this is the ability to divide an object or objects into a given number of like parts. It is this structure which concretizes this interpretation of the rational numbers. Thus, $x = 3/4$ means $4x = 3$, which can have the concrete meaning "x is the number we attach to each part which results when we divide three crackers into four equal parts."

As Freudenthal suggests, the system of rational numbers, especially the general algorithms, in this context cannot be understood without considerable algebraic practice. Fortunately, this is rather simple and is done as usual practice in junior high school mathematics. The principle base for developing rationals becomes solving equations of the form $a \circ x = b$, where a, b are integers and where "o" is usually interpreted to mean multiplication.

Again, one can ask, "What activities should occur prior to such equation solving?" If we look at the equation $3x = 6$, or better yet the equation $5x = 6$, we see the symbolic representation of the question: There are 6 pizzas for 5 children. What is an equal share for each? Here again, partition division activities come up as a preactivity to fraction or rational number work. The second equation is representative of partitioning continuous quantities--which is important in developing the notion of nonintegral rationals.

We can carry this activity back further in two ways. One is the partitioning of discrete sets evenly: Here are 20 letters; they are distributed evenly to 5 mailboxes. How many go in each mailbox? A possible algorithmic activity which relates to this form of division is a "dealing" activity. The other precursory activities to partitive division, particularly of the continuous variety, are those such as paper folding into congruent pieces. For example, the entertaining activity of "dragon curves" elicits appropriate behavior.

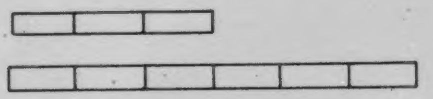
Here is a strip of calculator paper.
Fold it exactly in two.
Do this again with the folded strip.
And again.
Unfold the paper. How many parts do you have?________
What can you say about the parts?________



Here is a picture of a shape you can make. It looks like a duck.
Make as many shapes as you wish with your strip.

Of course, the important aspect of these kinds of activities is the process of dividing an object, particularly a linear one, into congruent pieces and the visual experience of seeing part-whole relationships. Coloring some parts red and others yellow would allow questions comparing the number of red with the total number, etc. This might well be an appropriate preactivity for the notion that $3 \times 1/8 = 3/8$. However, this is not truly related to the equation solving which is central to the quotient field factor of rationals. Nonetheless, the more or less free activity described above can be given a more explicit rational number format. For example:

Measure two strips of calculator tape. Make one 3 decimeters long and the other 6 decimeters. Mark off the decimeters on the strips.



Take the 3 decimeter strip.
Fold it evenly in two.
Fold it again.
Unfold it. How many parts do you have?________
How long is each part? $>$ 1 decimeter?________ $<$ 1 decimeter?________
Can you name its length in decimeters?________ decimeters.

Take the 6 decimeter strip.
Fold it evenly in two.
Fold it again.
Once more.
Unfold it. How many parts?____________
How long is each? > 1 decimeter?__________ < 1 decimeter?____________
How might you name its length?__________decimeters.
Tell something about the folded parts of the two strips.

Here the activity directly precurses the study of the equations like $4x = 3$ and $8y = 6$ and suggests their equivalence.

A major feature of the quotient field interpretation of the rationals is that the algorithms are derivable from equations via the field properties. There is no initial question of "common denominators" or their importance. By using the notion of measurement, one can provide the opportunity for concretely developing algorithms. For example, one could use all our old rulers divided into 16ths as "Fracto-Rules," that is, addition slide rules (see Figure 4).

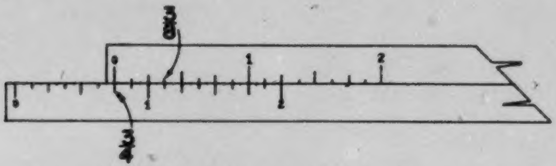


Figure 4. An example of addition slide rules.

Figure 4 illustrates $3/4 + 3/8 = 1\ 1/8$. There is no question as to whether adding is possible. After many such exercises, the student can see an algorithm which enables him to find the sum without measuring. A first step towards this would be a simple counting which would give $3/4 + 3/8 = 6/8 + 3/8 = 9/8$.

Experience with several such rulers broken conveniently into twelfths, eighths, or twenty-fourths would be background for the addition algorithm. In this case, the student would be deriving a concrete algorithm from the "axioms" of the setting. It should not be thought that such concrete settings will generate the general algorithm for adding fractions. It will serve as a basis for what Collis (1974) would call a "concrete generalization"--the organization of a variety of exemplars under a rubric. And, it certainly will provide considerable working knowledge with rational numbers.

In summary, the quotient field interpretation of rational numbers leads to notions of abstract algebra as well as the algebraic experience of domain extension. The fundamental prerequisite experience lies in the algebra of equation solving. A second fundamental prerequisite is the experience of deriving a definition for an operation from a setting. This second prerequisite is important because it is with the operations on rationals that a child has first experiences with an operation in the abstract, that is, one which must be defined and does not arise from simple intuition.

Prerequisite to both these experiences is work with partitions of continuous quantities. (As has been noted earlier in the paper, this also involves such notions as conservation of length and area).

The cognitive structures necessary for coping with this interpretation are those of formal operations, particularly the ability to generate implications. More primitive, or at least earlier, structures are those of concrete deduction and partitioning.

Rational Numbers as Measures

The measurement interpretation of rational numbers has been implicitly discussed in several earlier sections of the paper. In this part of the paper, rational numbers will be interpreted as points on the number line. Fundamental to this interpretation is the notion that the unit for the number line, once chosen, can be divided into any number of congruent parts. The rational number $a/b < 1$ then can be seen as a measure of a of b congruent parts.

Since the unit can be divided into any number of parts, adding simply means laying two vectors end-to-end and reading the result (see Figure 5).

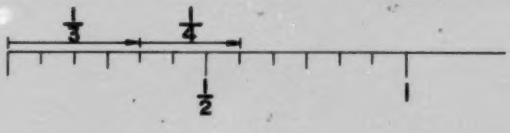


Figure 5. Addition of rational numbers as measures.

Thus, $1/3 + 1/4 = 7/12$. Of course the divisions of the units must be chosen so that it "works;" that is, the end of our second vector is on an exact division of the unit.

A similar development can be made for other operations. For example, in $2/3 \times 3/4$ we can break each fourth into 3 parts and count up 2 such parts for each fourth. Hence, we get six parts and only have to determine how these parts partition the unit (into twelfths).

The measurement interpretation of rational numbers is inherently an analytic interpretation--that is, relating to point sets. However, the notion of "flexible partitioning of the unit" allows the algebraic notions of operation and equivalence to emerge.

It should be noted that measurement is the natural locus for considering order. All of the order properties (e.g., trichotomy, antisymmetry, transitivity) arise naturally out of measurement situations. Previous sections showed that order could be given an "algebraic" interpretation. However, that interpretation was not intuitive and has limited instructional value.

Many of the cognitive structures necessary for coping with the measurement interpretation (e.g., partitioning or concrete deduction) have been discussed elsewhere. There are three such structures which seem particularly important here. The first is the notion of a unit and its arbitrary division. The child must realize that the unit is invariant under partitioning, and he must also realize that one can partition the unit into any number of congruent parts. Second, the child must be able to conceptualize part-whole relationships in this context and recognize equivalent settings arising from partitioning of the unit (1/2 = 3/6). Third, the child must develop the concept of an order relation. This involves both the ability to order physical reality and the ability to use correctly the symbolic order statements. Underlying these structures are more general structures, conservation of length and substance, and a general notion of ordinal number (see the paper by Brainerd).

Preactivities for the measure interpretation of rational numbers can be developed from both forms of division--measurement and partition. Using a unit of convenient length, say 24 centimeters, one can use various rods or strips to illustrate the variety of partitions. Using colors, one can measure 2/4 or 3/6 and vividly see equivalence (see Figure 6).

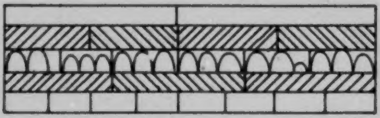
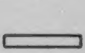
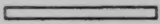
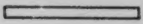
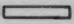
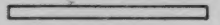


Figure 6. Various ways of partitioning a given unit.

Activities with a varying unit, measurement division activities, are also important in this development of the measurement interpretation of rational numbers. Questions given below illustrate this point.

If this  is the unit, then this  represents two.

If this  is the unit, what is this  ?

This  ?

Ordering of lengths is a natural activity, and the processes of comparing and seriating find numerous settings even for very young children. Such activities as described in the foregoing paragraph illustrate the fact that the measurement interpretation of rational numbers allows for direct contact with rational number concepts.

Rational Numbers as Decimal Fractions

A rational number is any number which can be expressed as a terminating or repeating decimal. This definition is the basis for the decimal-fraction interpretation of rational numbers. The mathematics for this interpretation needs little elaboration; the operations are just extensions of those for whole numbers, with division now not needing a "remainder." The numeration system covers the distinction between rationals and whole numbers both in terms of operations and the "ratio" aspect. If rational numbers were to be developed solely from this viewpoint, the prealgebraic experience with operations discussed earlier in the paper might be lost. In addition, pre-experience for rational expressions would not exist. However, rationals as decimals would serve as an adequate basis for analysis and would lead very easily to a discussion of irrationals. For the purposes of applied mathematics and computing devices, our set of rationals might even be limited to numbers expressible as 6 or 8 place decimals.

Rational numbers as decimal fractions are also the numbers of estimation. This is true whether one is measuring to the nearest millimeter or stopping an algorithmic search for zeroes of polynomials when $|x_{i+1} - x_i| < \varepsilon$. Rational numbers enter into numerical analysis problems and in the Cauchy sequence definition of real numbers.

The cognitive structures necessary for a child coping with decimals are similar to those mentioned for measurement. Here, however, there is more emphasis on ability to generalize in the symbolic domain. That is, the child must be able to generalize the decimal numeration system on its algorithms to this fractional situation. Also, the child must be able to attach a "standardized" description to part-whole situations. Thus, dividing an object into six congruent parts, taking one and saying "one of six" or "one-sixth" is a natural extension of counting; saying "about .16" is not. Thus, measuring and estimating are critical. The latter entails a general notion of unit and the ability to think hypothetically, even if about a real situation.

The activities from which decimal fractions proceed are in three areas. Rather obviously the representation and standard operations arise out of the decimal numeration system. Thus, all work with whole numbers and their operations with particular reference to numeration is critical. This work can lead directly to decimal fractions as seen in Frédérique Papy's work with the Mini-computer.

Measurement activities are a second obvious basis for decimal fraction consideration. However, with decimal fraction we divide the unit in a standard way, by successive powers of 10. A natural model for measurements which generate decimal fractions is the meter with the obvious subunits:

1 decimeter -- .1

1 centimeter-- .01

1 millimeter-- .001.

Of course, this model exists in other measurement modes as well, (e.g., capacity, mass, and money).

The third area of predecimal fraction activity is estimating. Statements in the "This is about so long." vein can be refined easily to illustrate decimal fractions as estimating numbers. The processes of seriating and comparing are of paramount importance as is the whole notion of order.

Rational Numbers--A Conglomerate

In the seven previous sections of this paper, different interpretations of rational numbers have been discussed. The fact that rational numbers have these diverse interpretations is by no means new. The work with unit fractions and applications in the pyramids by the Egyptians (number theory and proportions), the work of the ancient Greeks in commensurability (analysis), and the trigonometry of the Babylonians (decimal fractions) certainly bears this out. However, it is a major thesis of this paper that rational numbers, from the point of view of instruction, must be considered in all of the interpretations. From the point of view of curriculum, it has been common to implicitly assume that rationals had some single interpretation, and ideas were then developed within that one interpretation. This often meant that some rational number concept was difficult to learn (e.g., addition) or meant that some emphasis was deleted (e.g., algebraic aspect).

This singular rather than multisided viewpoint also effects the child who is learning about rational numbers. Because each interpretation of rationals relates to particular cognitive structures, ignoring a conglomerate picture or failing to identify particular necessary structures in developing instruction can lead to a lack of understanding on the part of the child. The conglomerate picture of rationals which will be charted will identify the complex cognitive structures which relate to (or form a foundation for) a child's idea of rational number.

As is seen in the earlier sections of this paper, the many interpretations of rational numbers have themselves many related instructional strategies. These in turn employ numerous physical and symbolic models. Without a conglomerate view, it is easy to design instructional settings which contain contradictory elements or models, or which do not easily lead to the development of some rational number concept. For example, if one interprets rationals as measures and uses a number line model, multiplication of rationals is not naturally generated. The number line model may conflict cognitively with an area model or an exchange model for generating multiplicative ideas.

Thus, a curriculum developer-instructional designer must ask what the short-term and long-range objectives are with respect to instruction in rational numbers. He must then ask how each of the interpretations relates to these objectives. He can then select particular interpretations in which to develop and reach certain objectives. Given these, he can then ascertain the necessary cognitive structures for meeting the objectives and develop sequences of instructional activities which contribute to the growth of these structures.

A researcher who asks, "How does the child know rational numbers?", must go through a similar process. He can study selected interpretations in more detail and identify what he believes to be the most important cognitive structures. Settings can then be developed or used which allow one to see the extent to which a child has such structures. The growth of such structures can then be studied developmentally. Alternatively, the importance of such structures can be tested. Here, one would test the effect of having or not having some structure on attaining some rational number objectives. In this case, care would have to be taken in order that the cognitive structure under consideration related to the interpretation used in the learning situation.

What follows is an attempt to present a more concise description of these seven rational number interpretations. For each interpretation, Figures 7-9 will contain indicators of the major mathematical structures and long term mathematical goals, the related cognitive structures, and the related instructional structures. Some structures which could belong to more than one interpretation will not be included in all of the figures. For example, computation with rationals is necessary in all interpretations but appears only under the "fraction" interpretation. The cognitive structure of proportionality again underlies any interpretation but will be charted only where it plays a most prominent role.

One step in using the conglomerate picture of rationals summarized in Figures 7-9 is to look at each interpretation to develop some network or sequence of instructional activities. This is perhaps simplest to do in terms of the mathematical structure. Because the cognitive structures relating to rational numbers are not part of the "natural thought" of the child in the same sense as more primitive number concepts, a network of these structures would probably not be satisfactory anyway. From this mathematical network, a derived network of cognitive structures can be developed. Finally, the network of instruction for each interpretation can be made.

Because the important cognitive structures within the fraction interpretation are skills with particular computations or comparisons, the hierarchy inherent in the mathematical structures would have almost direct parallels in terms of cognitive and instructional structures. As suggested earlier, there has been much work done throughout the century (as well as currently) in developing hierarchies of skills as illustrated in Figure 10.

Interpretation	Mathematical Structures and Directions	Cognitive Structures	Instructional Structures
Rational Numbers as Fractions	-equivalence fractions -skill with standard operations on fractions -word problem solving with fractions -skill with rational expressions -arithmetic + algebra with emphasis on skills	-appropriate hierarchy of rational number skills -concept of proportionality -transfer from one set of behaviors to another	-each operation analyzed into a sequence of behavioral objectives to be mastered; generally a rule (e.g., approach) -each operation followed by related word problems -detailed diagnosis of learning difficulties and related remedial instructional activities -prerequisite skills with whole numbers and equivalence determined
Rational Numbers as Equivalence Classes of Fractions	-ordered pairs -understanding rationals as equivalence classes of ordered pairs -symbolic control over part-whole relationships (fractional or ratio) -operations and properties of rationals (rationals as a field)	-ability to view reality in terms of coordinates -assigning symbolic representation to ordered situations -class inclusion -set correspondence -partitioning -proportionality -using equivalence to generate operations (informal deduction)	-concentration on developing a bank of experiences which will give meaning and understanding to later formal definitions; on a macro-scale (e.g., rule model) -experiences with equivalence and operations in state-state, state-operator, operator-state, and operator-operator settings to develop fractions (or ratios) as a symbolic means of comparing number attributes of part-whole situations -use of formal definitions and rules to summarize experience -corresponding pairs with whole numbers -dividing units arbitrarily -partitive division settings, discrete and continuous

Figure 7. Rational number interpretations and related structures.

Interpretation	Mathematical Structures and Directions	Cognitive Structures	Instructional Structures
Rational Numbers as Ratio Numbers	-ordered pairs -equivalence classes -multi-scaled number lines -operations on rationals	-symbolic control of proportionality -invariance of the number line under different "scales"	-dividing unit intervals in many ways -symbolically generating and comparing equivalence classes -defining operations in terms of selecting items from such classes
Rational Numbers as Operators or Mappings	-rational numbers as elements of an algebraic system -states and operators as basic mathematical elements -continuous representation, dilatations leads to transformation geometry or algebraic geometry -composition groups -arithmetic concepts derived from algebraic notions -essentially multiplicative	-comparing various structures -representing compositions as new entities -general notion of reversibility -preproportionality -proportionality -measurement division -abstracting properties from situations	-mappings (proportional) -exchange games -study of systems of operators (one exchange following another, one dilatation following another) -representing a combination or composition of operators with one operator -similarity activities -measurement division activities
Rational Numbers as Elements of a Quotient Field	-rational numbers as quotients -infinite ordered field -extension of domains to fields	-reasoning from hypothesis (implications in a symbolic form) -concrete deduction (given a situation derive properties) -proportionality -partitioning	-equation solving as the basis for formal algorithms -work with vectors on a conveniently divided number line to generate intuitive algorithms -arbitrary division of a unit -continuous partitive division activities -discrete partitive division

Figure 8. Rational number interpretations and related structures.

Interpretation	Mathematical Structures and Directions	Cognitive Structures	Instructional Structures
Rational Numbers as Measures	-rational numbers as points on the number line -operations as vectoral in nature -basis for analysis	-developing arbitrary division of the unit is central (a.d.u.) -comparing rational numbers by length -equivalence on the basis of unit division -operations on basis of arbitrary unit division -measurement division activities	-comparing sets and lengths -concrete generalizing -partitioning -comparing lengths -seriating -transitivity -ordinality
Rational Numbers as Decimal Fractions	-computations with decimal fractions -applications of rationals (e.g., metric measure) -estimation -reals as decimals	-decimal division of unit (d.d.u.) -use of decimal number line (e.g., meter sticks) to illustrate rationals -extension of operations on whole numbers -estimation activities	-symbolic generalizing -partitioning by 10 -comparing -ordering

Figure 9. Rational number interpretations and related structures.

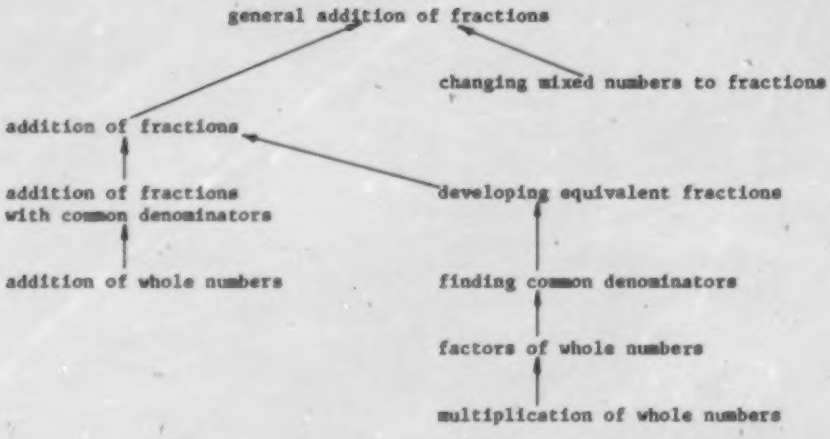

general addition of fractions
changing mixed numbers to fractions
addition of fractions
addition of fractions with common denominators
developing equivalent fractions
addition of whole numbers
finding common denominators
factors of whole numbers
multiplication of whole numbers

Figure 10. A skill hierarchy for the fraction interpretation of rational number.

However, within the other interpretations, there has been much less work. Also within other interpretations, the term hierarchy can only be used loosely; hence, the term network is used instead. In Figure 11, a very general network for the quotient field interpretation is given. The dashed lines indicate an important, but not necessary, path in the network. Alongside this network, the cognitive structures important to the mathematical structures are given. In a sense, these cognitive structures also form a network induced by the mathematical structures.

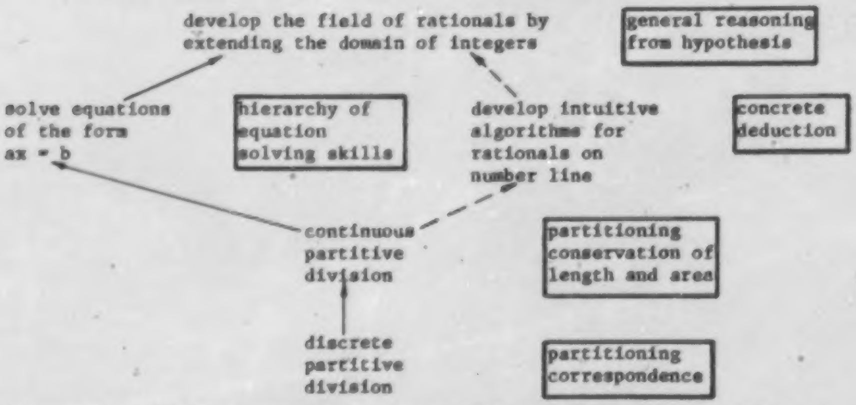

develop the field of rationals by extending the domain of integers
general reasoning from hypothesis
solve equations of the form ax = b
hierarchy of equation solving skills
develop intuitive algorithms for rationals on number line
concrete deduction
continuous partitive division
partitioning conservation of length and area
discrete partitive division
partitioning correspondence

Figure 11. A general network for the quotient field interpretation of rational number.

Clearly, considerable work is needed in elaborating and clarifying such networks for each of the interpretations.

As has been mentioned previously, work with these networks should prove fruitful. For the person studying the child's conception of rationals, the cognitive networks would be a source for hypotheses about children's behavior in terms of age or their position in an instructional sequence. This research might also be suggestive of a learning path which would be most congruent with a child's development.

The instructional designer would probably wish to try to form an overall network for rational number instruction. In studying the networks, he could identify elements of each which might fit into an overall instructional scheme.

In using the interpretations and related networks, two ideas need consideration. One pertains to the underlying mathematics, the other to instructional interpretation. Rational numbers, because they form a field, are both additive (in the measure sense) and multiplicative (in the mapping composition sense). Any research or development must take these independent notions into account. Further, physical models of rationals can be either discrete or continuous. The advantages and problems of each have been discussed above. The interpretations are related but not the same. The main feature of any continuous model is that it admits repeated and infinitely varied subdivision, while discrete models more readily admit counting as a strategy with less obvious emphasis on the unit.

The picture of rational numbers developed in this paper is necessarily a complex one. Rational numbers are seen to be algebraically significant and pervasive in many areas of mathematics. Thus, an instructional program which emphasizes one factor of this picture to the exclusion of others is inadequate. Many programs today still focus on the fractional and decimal interpretations. This ignores the algebraic notions inherent in the system or rationals and, in fact, leads almost nowhere. Similarly, a position which says "with metrics all we need is decimals" is ignoring much significant structural mathematics and applied mathematics which the rationals provide. It is further ignoring the contributions of other factors in understanding decimals.

The picture presented above challenges the mathematics educator to design a set of rational number experiences which provide the child with a balanced background. Although research is needed on this point, it seems sensible that the measurement and operator interpretations may represent early direct access to rational numbers, while the quotient field interpretation seems to represent a goal for later instruction. Further, it would appear that partitive division activities provide the experience base necessary for an understanding of the operation of addition. These and other fundamental questions need to be answered before truly adequate mathematics instruction can be developed for rational numbers.

Needed Research on Rational Number Learning

Any research on rational number learning must come to grips with three questions:

1. What is meant by a concept of rational numbers in childhood and adolescence?

2. How do these concepts develop?

3. What are instructional mechanisms that promote this development?

These three questions parallel the notions of mathematical, psychological, and instructional structures[1] developed earlier.

One direction for such research is seen in the work of Piaget on area and fractions (Piaget, Inhelder, & Szeminska, 1960). This work focussed on the young child's natural concept of fractional part based upon seven attributes: a whole composed of separable elements, that separation can occur into a determinate number of parts, that such subdivision exhausts the whole, there is a fixed relationship between dividing cuts and subdivisions, that the subdivisions are equal, that these parts are also wholes in their own right, and the whole is conserved (pp. 309-311). These attributes were studied across several configurations and denominations of fractional parts. This work and that reviewed by Payne (see the Payne paper in this collection) suggests that children well into elementary school do not attain the concept of fractions as defined above. Yet, this concept itself is very limited if one is considering a conception of rational numbers. The Michigan studies reviewed by Payne add the following rational number properties: attributed symbolic control over fractions (e.g., relating words, pair symbols, diagrams and physical instances of fractions), continuous and discrete part-whole relationships, rational numbers whose absolute value is greater than one and equivalent fractions. In addition, these studies also added the notions of operations and order to the concept of rational numbers.

The conglomerate model suggested earlier identifies seven interpretations which again enlarge on the concept of rational number. The number of "attributes" added to the concept is too long to elaborate. Such "primitive" attributes as arbitrary subdivisions of a unit, repeated standard subdivision (decimals), some aspects of the proportionality schema, and concrete deduction with fractional phenomena are important under the various interpretations of rational numbers. The more formal "attributes" of general one-dimensional vector addition, composition of function, deductive reasoning with multiplicative equations, and applying rational numbers as measure estimates are all seen as part of a mature concept of rational number.

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[1] The genesis of these ideas came from a discussion with Richard Lesh.

Suffice it to say that the answer to question number one is very complicated, and research questions two and three are at least as complex. The remainder of this paper will be devoted to the very brief elaboration on two models seen as useful in pursuing these questions. An attempt will be made to suggest general research areas and relate these to previous research concerning rational numbers as well as research directions posed in the area of measurement by Carpenter and Osborne (in this collection).

A Curriculum Research Model and Its Implications

Because it seems that the complex concept of rational number demands that it be, to some extent, formally learned, one model useful in focussing research is a curricular one. Before moving specifically to rational number considerations, let us define curriculum research as it will be viewed in this section. Romberg and DeVault (1967) see mathematics curriculum as the total school mathematical experiences of the child--with input from the learner, teacher, instructional procedures, and mathematics programs. In this section, we will emphasize the latter two input sources into a system of instruction in rational numbers. Romberg and DeVault (1967, p. 108) lay out the steps in developing such a system (see Figure 12). The companions to this paper are efforts in the "analysis" stage of this model, with the work of Payne, Muangnapoe, Green, Williams, Choate, Galloway, and Ellerbruch being examples of methodologies appropriate for the "pilot" stage (see the Payne paper in this collection). Although well developed concepts of rational numbers have been known for thousands of years and much research in this century has been devoted to aspects of rational number learning, the work of Payne et al. suggests that much research needs to be done at the analysis and pilot stages of the Romberg-DeVault model.

The seven interpretations of rationals also suggest a comprehensive instructional analysis. A major question is "Can a mathematics program be developed which appropriately emphasizes each of the interpretations of rational numbers?" In most previous research, this particular question has been ignored. This has led to programs in which only certain objectives within the study of rational numbers were reached or even considered, while others were ignored. There is a need to develop a set of objectives for rational number learning, test this set against the seven interpretations to see if it is comprehensive, and to relate it to the outcomes of instruction in other aspects of mathematics learning. For example, if the

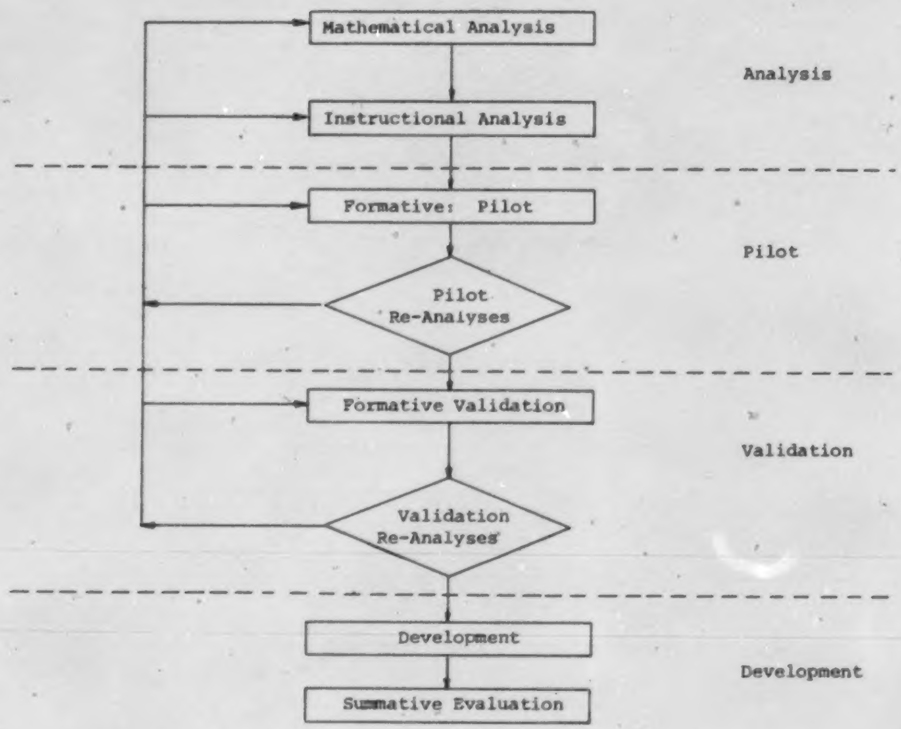

Figure 12. Romberg and DeVault's steps in developing an instructional system.

operator interpretation is ignored under rational numbers, are the mathematical directions of this interpretation developed under some other mathematics objectives set (e.g., integers or transformation geometry)? If the fraction interpretation is de-emphasized, are these outcomes important and, if so, are there compensatory objectives available in other rational number interpretations or topics?

A related need is to see if instruction under one interpretation of rational numbers conflicts with instruction under other interpretations. For example, Payne, in his review of research on rationals, conjectures such a conflict between a set based approach (ordered pair interpretation) and measurement. A well developed instructional analysis could point out potential conflicts in the effects of instructional activities. For example, it seems logical that the operator interpretation would conflict with the objectives related to adding rationals. Conjectures of conflict could be carefully tested in a pilot context. Where such conflicts existed, sequences could be developed which would minimize such conflicts, while maximizing the comprehensive nature of the rational number experience program.

The mini-max problem leads to sequence design research. A central question here is: "Should certain rational number interpretations be emphasized at certain stages in the instructional sequence K-9?" A second question is: "Are there appropriate mixes of instructional procedures which can be used at various stages in an overall rational number program?" Specifically, a study could attempt to predict analytically the best mix of instructional procedures useful at the earliest stages in the sequence. These mixes could undergo a careful pilot analysis with children at various ages from 6-9.

Another form of curriculum research is longitudinal studies. Clearly, such studies pertain to the validation of any balanced rational number program suggested above. Such studies could help answer other questions, however. Programs based on decimals; measurement; measurement, operators, and quotient fields; decimals and quotient fields suggest themselves as amenable to longitudinal study.

Specifically, with the coming of the metric system, there has been discussion of the increased emphasis on the decimal interpretation. It would be interesting to study the effect of a curriculum based on the decimal interpretation of rationals. Over the last 50 years, there have been numerous studies on grade placement which have suggested a postponement of fraction work until junior high school, with an earlier introduction of decimals. Thus, there is evidence for the feasibility of such a curriculum. However, numerous questions remain unanswered. What would be an appropriate instructional structure with which to introduce decimals? A number line, a rectangular region, a "hundredths" frame, or a meter are all possibilities. More generally, what "fraction" background facilitates decimal work? How does the ability to grasp the repeated (by a factor of 10) division of a unit relate to decimal learning? Because computation with decimals is an easy symbolic generalization from whole numbers, to what extent are manipulative representations useful and necessary? What is the role of calculators in such a curriculum, and how is decimal learning enhanced by their use? If the decimal interpretation is only taught through grade 7 or 8, what are the limitations on the students' concept of rationals (e.g., algebraic aspects, part-whole relationships, or proportionality)?

There are many studies suggested by these questions. One could carefully design a sequence for grades 4-6 based on the decimal factor and follow a sample of children through it. They could be tested on various aspects of rational number behavior (see following section) and could be given general rational number knowledge tests periodically. Such a study would have to follow the rubric of the curriculum model.

General Rational Number Research Implementation Model

The Romberg-DeVault model described in Figure 12 is limited because it only pertains to the second of the initially posed questions. That is, although the curriculum research model assumes knowledge of the development of a concept, it does not contain a component which attends to the study of the rational number behavior of children and adolescents. Because the concept of rationals is complex and because numerous instructional structures and representations suggest themselves, the child's reaction and interpretation of such representations and structures is an important component of rational number learning research. Again, the Romberg-DeVault curriculum model does not explicitly relate to this general problem.

A more general model or organizer is suggested by Sawada and Cathcart[2] and is illustrated in Figure 13.

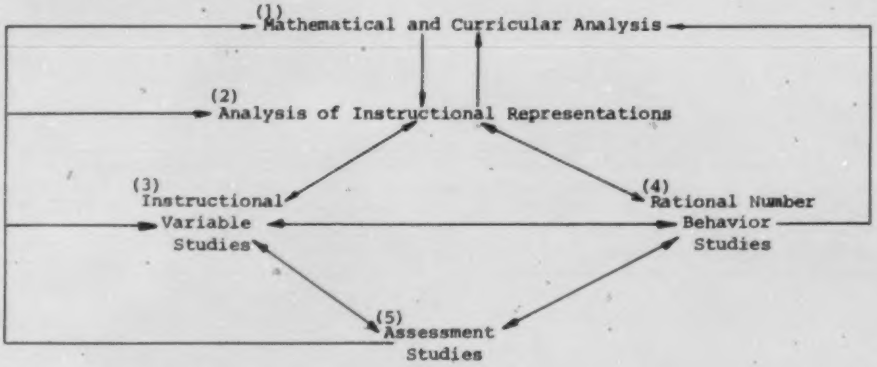
(1) Mathematical and Curricular Analysis
(2) Analysis of Instructional Representations
(3) Instructional Variable Studies
(4) Rational Number Behavior Studies
(5) Assessment Studies

Figure 13. The Sawada and Cathcart curriculum development model.

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[2] This idea was developed at the Workshop on Number and Measurement sponsored by the Georgia Center for the Study of Learning and Teaching Mathematics.

A two dimensional representation of this model is misleading in that it suggests an ordered sequence. The arrows should not suggest an ordering, but a flow of information among research on various problems.

The earlier sections of this paper and the Payne paper discuss research in area (1) of the Sawada-Cathcart model. In a sense, any research devoted to answering the question, "What is the concept of rational numbers?", would fall in this area.

The complex notion of rational numbers discussed above was explicated through the use of, and in some ways developed from, a wide variety of instructional representations. The instructional procedures suggested for rational numbers all made use of one or more instructional representations. One set of possible studies would consider children's reaction to such representations. That is, such studies would attempt, on a clinical basis, to ascertain just what it is that a child sees when faced by a particular representation. Such research is prompted by the problems that children may have had in working with discrete models of fractions (see the Payne paper). It may turn out that particular representations do not convey the desired information to the child.

In the Payne paper there are details of a series of on-going Initial Fraction Sequence studies (IFS) at Michigan which could be classified as instructional variable studies. And, because measurement and measure concepts (unit, area, etc) are closely related to early instruction on rational numbers, it is also important to study the relationship between development of measure and rational number concepts. For example, one such study would try to predict success under IFS instruction from the student's concept of area.[3] A similar study could investigate relationships between measure concepts and rational number learning under a number-line treatment. Another study could consider the sequencing of instruction on measurement and rational numbers which might answer the question: "Is measurement study a prerequisite to rational number study."

The curriculum and instructional variable questions mentioned previously relate to, and in many cases hinge on, the ability of children to perform or acquire certain rational number behaviors. The paradigm for this work is perhaps best illustrated in the work of Inhelder and Piaget (1958) on the growth of logical thinking. Like the concept of rational number, the concept of logical thinking is a complex phenomenon. Piaget and Inhelder identified behavioral correlates of aspects of logical thinking and then posed critical tasks eliciting these behaviors from children from ages 5-15. The purpose was to verify, illustrate, and detail the stages of the growth of logical thinking. A similar study of the growth of rational number thinking is suggested below. Some aspects and behaviors of rational number will be impossible to study in their "natural state." They will undoubtedly be colored by instructional experience. It is hoped that the behaviors posed for study below will be less affected by schooling than those regarding function and proof discussed by Lovell (1971a, 1971c).

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[3] Suggestion for some of the studies were made by members of the "Rational Number Working Group" at the Workshop on Number and Measurement sponsored by the Georgia Center for the Study of Learning and Teaching Mathematics. Members of this group were: E. Begle, G. Cathcart, R. Ceileski. M. Herman, D. C. Johnson, R. Kalin, T. Kieren, J. Kirkpatrick, D. Owens, J. Payne, E. Rathmell, and H. Wagner.

Relatively little data is available on relevant behaviors such as discrete and continuous partitioning, apart from that presented by Piaget et al. (1960). This is evidenced in the various pilot studies described in the Payne paper (in this collection). In a study concerning the behavior of boys in grades 1-3, Little (1974) found that in discrete measurement and partition division settings there was very little partitioning behavior, even in obvious partitive settings. (For example, here are 15 cars, park them in 3 lots so the number in each is the same.) Thus, research such as the following is suggested.

Three discrete and three continuous partition division settings could be developed. These settings would involve the child in physically solving some problem. Samples of children at age levels from 5-12 could be tested in some or all of the settings. A well developed video-recording protocol à la Nelson and Sawada (1975) could be used to collect the data. If this were not feasible or desirable, some form of experimenter-observer team could be employed, as well as using written records made by the children. This data could then be carefully analyzed to develop a set of behaviors generated by children which might be suggestive of some developmental trend. This experiment could also be done with measurement activities (discrete and continuous), with equivalence problems, and with exchange (m-operator) problems. Using older subjects (perhaps ages 10-15), more symbolic problems would be appropriate. These might include settings involving arbitrary division of a unit, solutions to quotient equations, reacting to graphs of equivalence classes, ratio problems, and quotient problems. As suggested previously, one goal of such research is to develop some form of catalogue for rational number behaviors.

Since these behaviors are important to rational number learning, instructional protocols aimed at developing specific behaviors could also be researched. For example, are there protocols useful in developing continuous partitioning behavior? At what age do such protocols seem optimally effective? Such questions need careful research based on all four phases of the Romberg-DeVault model.

The conglomerate model speculates on processes which seem necessary to learning within a particular interpretation of rationals. Both further analysis and experimental study are needed to uncover relationships between such processes and the rational number behaviors suggested above. For example, it would seem that conservation of area and length might be related to continuous partitive division.

The conglomerate model is also suggestive of relationships between rational number behaviors and learning of particular facets of rational number knowledge. Again, research is needed to test the validity of such suggestions. For example, does work with continuous partitioning problems promote learning of equivalent fractions or of solution to quotient equations (e.g., $ax = b$)? Payne et al. have made use of methodologies appropriate to the study of such questions. Pilot and validation studies should be conducted to answer questions such as those above. These studies would also include a review of research findings trying to use known conclusions to test relationships such as those suggested.

The last category of studies mentioned in the Sawada-Cathcart model are the assessment studies. There have been a large number of studies done over the last 50 years which have tested the child's rational number knowledge of achievement. Yet, many of these have contained mainly analyses of children's computation with fractions. Studies need to be done with large samples of children using tests based on a more complex view of the concept of rationals. One such study might look at the attainment of above average 14-year-olds who have undergone what is considered to be excellent instruction. This study might give an upper-bound estimate of the effects of current instruction on rationals. Another study might assess the rational number knowledge of large samples of pre-adolescents at several age levels. A third study might look at low achieving high school students and assess their rational number knowledge.

Because of their important effect, a study of teachers' concepts of rational numbers, based on a complex view of this concept would be enlightening. While this could most easily be done with preservice teachers for both elementary and secondary schools, a study involving a broad range of inservice teachers would be most interesting.

Summary

Several kinds of research on rational number learning have been suggested. The analytic curriculum research represented by the companion papers to this paper need continuation, with the conglomerate model and pilot work cited as stimuli. Under this rubric, both pilot studies and longitudinal effect studies are suggested.

In developing the studies suggested above and in developing curricula based on them, knowledge of effects of rational number behaviors of children is needed. Clinical studies of such behavior are suggested. In addition, experimental studies on the learning of such behaviors are needed. To adequately tie such behaviors to a curriculum, they need to be related to various psychological processes and to specific facets of rational number learning.

If such studies are to really bear fruit, there must be cooperation and coordination among them. It should be reasonable that the rational number behavior studies, the instructional studies, the representation studies, and assessment studies could share common tests, items, or methodologies. As information becomes available from one study, it should inform and indeed change other further studies. With such interaction, an adequate answer to the three questions posed initially can be found.

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Review of Research on Fractions

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Overview

This report of research on fractions is organized chronologically by studies done at Michigan since 1968. Other research is included as background for these studies. This organization is intended to convey the shift in direction of research and in questions asked over an eight year period of work in this one content area. To show this shift in direction, more detail is given to the Michigan studies. However, the studies reported in less detail are by no means less important. Space limitations preclude full discussion of all studies.

The Michigan studies done from 1968 to 1973 examined various approaches to fraction algorithms and various manipulative materials. The first study was done in 1968 by Bidwell on division of fractions, comparing three approaches to the operations. Green (1969) compared two approaches to multiplication of fractions as well as the use of diagrams vs. manipulative materials. Bohan (1970) compared two different sequences involving equivalent fractions and two types of materials. Coburn (1973) studied two approaches to equivalent fractions: one using ratio and one using regions. While each of these studies examined learning sequences and identified weaknesses and strengths in children's cognitive structures, emphasis was placed on comparing different algorithms for a given fraction operation.

Beginning in 1973, emphasis shifted from comparing various algorithms to a more intensive examination of what children learn while being taught a carefully developed sequence. The investigators were influenced by Bloom's interesting work on mastery learning and by Greeno's exciting research on problem solving using information processing models. Much pilot work was done, trying to develop sequences with mastery learning as a goal. At the same time, serious effort was devoted to carefully assessing the way the learner's cognitive structures were developing.

Muangnapoe (1975) focused on the learning of initial fraction concept and symbols. Williams (1975) examined the learning of initial fraction concepts in a Detroit inner-city school. Galloway (1975) taught

Muangnapoe's initial fraction sequence to children's in grades one through five and examined the developmental patterns from grade to grade; she also taught beginning decimal concepts to children in grades three through five. Choate (1975) examined the cognitive structures of children who were taught a rule for comparing fractions vs. children who were taught a conceptual approach to comparison. Ellerbruch (1975) studied the effects of changing the order of rules and concrete models for equivalent fractions, and for addition and subtraction.

Thus, the Michigan studies have shifted in emphasis. They began by comparing different algorithms, with an analysis of the strengths and weaknesses of each. Recently, they have been more developmental. More pilot work is being done as instructional materials are developed. Currently, however, more attention is being given to the cognitive structures resulting from different combinations of rules and concrete models.

A Note About Language for Fractions

There is great variety in the language used in connection with fractions. For example, in the Michigan studies, "fractional number" and "fraction" are used almost interchangeably. When particular emphasis is needed for the symbol, the words "fraction symbol" have been used. Most arguments about language seem to rest eventually on the personal preference of the person advancing the argument. And, it is highly likely that having stated a personal preference in a general way, a contradiction emerges shortly thereafter in specific ways. For example, after saying that "fraction" is a symbol, the person is likely to use "fraction" to mean either the number or the ordered pair of whole numbers for the fraction. The question "What fraction of the pie is left?", "How do you add fractions?", "What fractional part of the inheritance is mine?" all reflect the idea of number than that of a symbol. In this report, usage of terms should make their meaning clear.

Division of Fractional Numbers (Bidwell, 1968)

In the mid-1960's, much emphasis was placed on a correct mathematical development of topics in school mathematics, including those topics taught in the elementary school. Division of fractional numbers, often considered the most mechanical and least understood topic in elementary school, was subject to a major new approach. This approach, labeled Inverse Operation by Bidwell, was compared with two other algorithms called Complex Fraction and Common Denominator.

The steps in each algorithm are as follows:

Common Denominator (CD)

$$a/b \div c/d = ad/bd \div bc/bd = ad \div bc = ad/bc$$

Complex Fraction (CF)

$$a/b \div c/d = \frac{a/b}{c/d} = \frac{a/b \times d/c}{c/d \times d/c} = \frac{a/b \times d/c}{1} = a/b \times d/c = ad/bc$$

Inverse Operation (IO)

$$a/b \div c/d = \square \Rightarrow \square \times c/d = a/b \Rightarrow (\square \times c/d) \times d/c =$$
$$a/b \times d/c \Rightarrow \square \times (c/d \times d/c) = a/b \times d/c \Rightarrow \square \times 1 =$$
$$a/b \times d/c \Rightarrow \square = a/b \times d/c \Rightarrow \square = ad/bc$$

Bidwell used 21 intact sixth grade classes (n = 448) in Saginaw, Michigan. There were eight lessons for each sequence, a review session, and a final testing. Daily criterion quizzes were given. Bidwell compared achievement on concept attainment related to the algorithm, division of fractions, division with mixed forms, and applications and multiplication of fractions. Retention was measured three weeks after the posttest. He also did a Gagné-type analysis of the learning structures for each algorithm.

Two analyses of covariance were run: one using arithmetic achievement as a covariate and the other using IQ as a covariate. Adjusted means are reported in Table 1, using arithmetic achievement as the covariate. The results were very similar using IQ as the covariate.

Table 1 shows that there was consistent superiority of the Inverse Operation treatment, although much of the advantage was lost on the retention test. The Complex Fraction treatment held a decided edge in retention rate. The Common Denominator treatment was consistently inferior to either of the other treatments, even for low ability students.

Lower achievement in the Common Denominator treatment was probably caused by the poor internal cognitive structure which the children developed. On the Gagné-type analysis of the internal structure of each algorithm, Common Denominator was clearly inferior with the two other treatments comparable.

Correlations were run between skill in division of fractions and scores obtained by testing the various parts of steps of a given algorithm. As Gagné had suggested in his work, correlations were positive but relatively low: Common Denominator, r = .60; Complex Fraction, r = .45; and Inverse Operation, r = .51.

Poorly learned content. There were some major content areas not learned well by pupils. Finding common denominators was difficult for the CD group. Equivalent fractions were troublesome for CD and CF. Interpreting a fraction as an indicated division was hard for the CF pupils. Relating multiplication and division sentences was a difficult task for pupils in IO.

Table 1

ANCOVA
Post- and Retention Tests on Division of Fractional
Numbers: Arithmetic Achievement as a Covariate (Bidwell, 1968)

Test	Posttest			Retention Test		
Treatment[a]	CD	CF	IO	CD	CF	IO
Percent total concepts attained (related to given algorithm)	69(23)[a]	70(26)	80(19)**	65(25)	66(26)	83(17)**
Part a. Division of fractions: (six possible)	4.5(2.1)	4.1(2.2)	4.7(1.7)**	2.6(2.2)	3.0(2.3)	3.0(2.4)
Part b. Division with mixed forms (transfer): (four possible)	2.0(1.5)	1.6(1.5)	2.2(1.4)**	1.3(1.6)	1.5(1.5)	1.7(1.6)
Part c. Applications (transfer): (three possible)	0.7(0.9)	0.9(1.0)	1.4(1.2)**	0.4(0.8)	0.6(0.9)	1.0(0.9)
Total score (thirteen possible)	7.1(3.9)	6.6(3.9)	8.3(3.6)**	4.3(4.2)	5.1(4.2)	5.6(4.6)
Mulitiplication of fractions (twelve possible)	7.2(4.6)	7.2(4.6)	8.7(3.7)**	7.1(4.7)	7.6(4.5)	8.6(3.8)**
Retention rate (comparison of retention test to posttest)						
Part a	58%	74%	63%			
Part b	69%	90%	76%			
Part c	59%	63%	68%			
Total	61%	76%	67%			

[a] Standard deviation in parentheses.
*$p < .05$
**$p < .01$

Related research. In past research where meaningful approaches to operations on fractions have been compared to mechanical or rule approaches, there appears to have been some advantage for the ones that were meaningful (Alkire, 1949; Brooke, 1954; Capps, 1960; Howard, 1947; Krich, 1964). Furthermore, when there was an advantage favoring meaningful approaches, it was usually most evident on retention tests.

Studies by Capps (1960) and Sluser (1962) dealt specifically with division of fractions. In the Capps study, the "common denominator" method was compared with a "rote inversion method." Division skills were comparable for both groups, but the inversion group did better on multiplication skills. The common denominator method seemed to produce some confusion on multiplication of fractions. Sluser used two different sequences for dividing fractions. The first sequence used a common denominator approach followed by the complex fraction method. The first sequence used a textbook, but the second used lessons designed by Sluser. The advantage at the end of instruction was for the pupils in the first sequence, but this was lost by the time retention tests were given. Bidwell conjectured that one of the difficulties with the second sequence resulted from interference between the two rationalized methods.

Comments. There were problems with many of the studies on operations with fractions that Bidwell tried to remedy in his own study. In the other studies, "meaningful" and "mechanical" were not defined clearly. Furthermore, other studies and methods books that were examined often confused the development of an algorithm with the statement of the rule. For example, "inversion" was often cited as a method as if it had been developed in a meaningful sequence of steps when it was nothing more than a rule children were taught.

Bidwell developed three algorithms, all meaningful, and he specified carefully what was involved with each. Each step in the algorithms was developed with care, and the final rule was designed to grow from the development and not be something separate.

From Bidwell's study, it was learned that it is important to do an analysis of the various components of the learning structure for an algorithm. Poor performance on some major subskills needed in the development of an algorithm showed spots where more effective instructional materials were needed.

The biggest surprise in Bidwell's study was the poor performance of low ability pupils using the Common Denominator method. Methods books for many years have recommended the common denominator approach as easiest for slow learners. However, Bidwell's analysis revealed that the Common Denominator method lacks internal structure compared with the Complex Fraction or Inverse Operation methods. That is, the Common Denominator method did not hang together logically as well as the other two methods. Bidwell demonstrated that an algorithm with good internal structure is just as important for slow learners as for their brighter classmates.

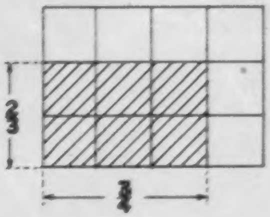
While Bidwell did not examine mastery learning per se, the mean score on the posttest was less than 80%; the score on the retention test was only about 50%. None of the mean scores on any of the algorithms even approached mastery learning levels for the simple skill of dividing fractions. Consequently, it would be interesting to know how much Bidwell's nine-day instructional sequence would have to be expanded and extended to produce mastery learning and how this would affect retention.

For the Complex Fraction and Inverse Operations sequences, there was reliance mainly on a logical development with a logical rationalization of each major step. There was minimal use of diagrams or concrete objects. In the Common Denominator sequence, analogies were made with concrete measurement situations and diagrams were used. Bidwell did not, however, investigate the relative effectiveness of concrete materials. This was done in a study on multiplication the following year.

Multiplication of Fractional Numbers (Green, 1969)

Green's background as an elementary school teacher and supervisor led her to consider algorithms for multiplication which had a logical development but which relied heavily on physical representations. She used a two by two design to investigate the effects of concrete materials vs. diagrams and an area model vs. a fractional part model.

Area. In the area models, a rectangular region was used to develop an algorithm for multiplication which extended an idea developed for multiplication with whole numbers. For example, 2/3 x 3/4 = n is illustrated by the shaded area in Figure 1. Since Area = length x width, the area is 2/3 x 3/4.



$$\text{Area} = \frac{2}{3} \times \frac{3}{4} = \frac{2 \times 3}{3 \times 4} = \frac{6}{12}$$

Figure 1

Finding fractional parts of (Of). The fractional parts model has been the traditional model used. Multiplication is developed using the first factor as a multiplier to find a fractional part of a region. To illustrate 2/3 x 3/4 = n using this approach, one begins with 3/4 of a region (see Figure 2a). Then the number 2/3 is used as an operator and two-thirds of the shaded region is found (see Figure 2b).

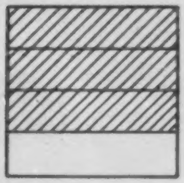


Figure 2a

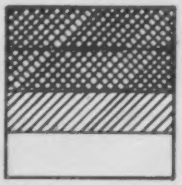


Figure 2b

$$\frac{2}{3} \text{ of } \frac{3}{4}$$

$$\frac{2}{3} \times \frac{3}{4}$$

Concrete. One inch paper squares were used as manipulative material.

Diagrams. Pictures were drawn for the algorithm but the squares themselves were not manipulated.

Green used 20 intact fifth grade classes (n = 481) in Waterford, Michigan with four teachers per treatment. There were 11 lessons and a posttest. A retention test was then given three weeks later.

Table 2 contains some data from her study. The mean scores were adjusted, using arithmetic achievement as a covariate.

Green (1968, pp. 186-188) summarized her results as follows:

1. The Area approach was better for learning multiplication of fractional numbers than the Of approach.

2. Diagrams and manipulative materials were equally effective in learning multiplication of fractional numbers.

3. The treatments were ranked in order of superiority in learning multiplication of fractional numbers as Area-Diagram, Of-Materials, Area-Materials, and Of-Diagram.

4. None of the treatments was best for the high achievement groups to learn multiplication of fractional numbers, and the Area-Diagram treatment was best for middle and low achievement groups.

5. The Of-Materials treatment was the least effective for low achievement groups to learn multiplication of fractional numbers.

6. With the exception of the Area-Diagram girls, there were no differences between the boys and girls in learning multiplication of fractional numbers.

Table 2

ANCOVA
Post- and Retention Tests on Multiplication of Fractional Numbers: Arithmetic Achievement as a Covariate (Green 1969)

Test	Posttest		Retention Test	
Model[a]	Area	Of	Area	Of
Concepts related to algorithm (fifteen possible)	10.8(3.2)**[a]	9.5(3.1)	9.3(3.3)**	8.5(3.4)
Computation (twelve possible)	9.5(2.4)*	9.0(2.6)	8.2(2.4)	8.0(2.9)
Application (twelve possible)	7.8(2.2)	7.5(2.2)	7.4(1.6)	7.5(1.7)
Posttest total (thirty-nine possible)	28.1(6.2)**	26.0(6.6)	24.9(6.0)	24.0(6.7)
Transfer (fourteen possible)	2.6(3.2)	2.9(3.6)	—	—
	Diagram	Concrete	Diagram	Concrete
Concepts related to algorithm (fifteen possible)	11.2(2.8)**	9.1(3.3)	9.5(3.2)	8.4(3.5)
Computation (twelve possible)	9.1(2.5)	9.4(2.4)	8.0(2.5)	8.2(2.7)
Application (twelve possible)	7.6(2.2)	7.7(2.2)	7.4(1.7)	7.5(1.7)
Posttest total (thirty-nine possible)	27.9(6.3)**	26.2(6.6)	24.8(6.2)	24.1(6.5)
Transfer (fourteen possible)	2.6(3.1)	2.9(3.7)	—	—

[a] Standard deviation in parentheses
*$p < .05$
**$p < .01$

7. The Of group had more favorable attitudes toward multiplication of fractional numbers than the Area group.

8. The Diagram group had more favorable attitudes toward multiplication of fractional numbers and their instructional aids than the Materials group.

9. The Of-Diagram group had the highest attitudes of the treatment groups, and the Area-Materials group had the lowest attitudes.

10. Both the Area-Diagram groups and Of-Diagram groups liked using diagrams significantly better than the Area-Materials and Of-Materials groups liked using manipulative materials.

11. An analysis of the learning structure is valuable in determining weaknesses and planning a correction of these weaknesses.

12. All groups had difficulty in finding a fractional part of a set.

13. The Area-Diagram and Area-Materials groups had difficulty with the Area models for multiplying a fractional number by a whole number and for multiplying a whole number by a fractional number.

14. The Of-Diagram and Of-Materials groups had difficulty with the Of models for multiplying a fractional number by a whole number and for multiplying a whole number by a fractional number.

15. All four groups had difficulty with renaming a number expressed in mixed form as a fraction.

16. All four groups were able to multiply a fractional number by a fractional number.

17. The multiplication of fractional numbers expressed in mixed form was difficult for all groups.

Green noted further:

> The failure in finding a fractional part of a set definitely points to the need to find a more effective way to teach this important concept. Particular attention should be given to overcoming the difficulty children have with the "unit" idea, relating the model and the procedure for finding a fractional part of a set, and delaying the rule until there is understanding of the concept.
>
> The difficulty with renaming numbers expressed in mixed form as fractions indicates a need to investigate an improved way to teach this. In addition to using models to develop understanding of this important concept, the study should investigate a way to relate the model with the procedure. Pupils should understand why the procedure works before using it. (p. 188)

The Gagné-type analysis of the learning structures showed a higher percentage of negative transfers within the structure for the groups using manipulative materials. Green conjectured that this was caused by difficulty in transferring from concrete objects to diagrams.

Related research. Most studies on multiplication of fractions are similar to those on division. They involve assessment of various meaningful or drill approaches, analysis of errors pupils make on multiplication, and, more recently, effects of programmed materials and various reinforcement schedules.

The study closest to Green's was done by Howard (1947). He used three "methods" for teaching a wide variety of fraction topics in grades five and six, including multiplication in grade six. He called his methods Drill, Meaning, and Combination. By Drill he seemed to mean giving children rules to follow and practice in computation--with no developmental work or concrete models. By Meaning he meant the use of concrete models and visual aids. The Combination approach was described as including both Meaning and Drill. Howard found that none of the three methods was superior at the end of instruction, either at the end of grade five or grade six. He did find, however, that at the beginning of the next school year pupils in the Meaning and Combination groups far surpassed the Drill group. Thus, on long term memory, the Drill approach was demonstrated to be inferior.

Several studies involved preparation of different instructional materials and various feedback or reinforcement schedules. For example, Miller (1961) and Triplett (1962) investigated multiplication. Miller designed materials with prompt feedback for pupils and found them superior to the "regular text" on a posttest. No retention test was involved in Miller's study. Triplett designed special materials for high, middle, and low ability pupils in grade six, attempting to adapt to individual differences among the three groups. He compared achievement on his materials with achievement using the "regular text." Triplett found his materials superior for the middle and low groups and concluded that differentiated written materials should be provided.

Arvin (1965) found no differences in achievement between children using programmed materials compared to those in "regular classroom" instruction. However, the programmed materials took only half the time. Furthermore, using programmed materials, Austin (1965) found no differences between treatments comparing multiple-choice answers vs. written answers and 100% reinforcement vs. 50% reinforcement.

Morton (1924) and Brueckner (1928) analyzed errors pupils make on all operations with fractions. Fewer errors were found on multiplication than on the other operations. Other major sources of errors were: (a) performing the wrong operation, (b) failing to reduce to lowest terms, and (c) making mistakes in computation.

Comment. As in the study by Bidwell, Green was more specific in designing and describing her sequences. She avoided the language of "Drill Approach" or "Meaningful Approach" evident in the literature and in other studies. All her approaches would have been considered meaningful, although they are nearer to Howard's "Combination" than to his "Meaningful." Green mixed meaningful work with practice more than Howard probably did.

It is important to note that a "Drill Only" or a "Rule Only" approach, an approach lacking either a mathematically logical development or concrete representation, was never used in any of the Michigan studies. (In later studies, Choate and Ellerbruch have investigated different ways to teach the rules for computation. Results from Choate's work are reported later in this document.)

All of Green's approaches involved some visual model: either concrete materials that children manipulated or diagrams of regions. Looking at her data, the most striking result is that her retention scores for any of the four treatments were almost 90% of the posttest score. Her retention rates were much better than the ones obtained by Bidwell (61% to 76%). Thus, it appears, in retrospect, that the use of visual materials in developing algorithms has a more important effect on retention than does a purely logical mathematical development such as the ones used by Bidwell.

While none of Green's approaches had a significant advantage on the retention test, it is striking that the cognitive structures for pupils in the Area approach were superior to those in the approach Finding Fractional Parts. Since there are more applications and practical uses for finding fractional parts, what may be needed is an investigation of how this approach can be taught so that the cognitive structure has a better fit than it seemed to have in Green's study. Furthermore, a careful analysis is needed of the differences that emerge when one approach is used to develop an algorithm and then another is used to teach application of the algorithm. For example, in using an initial approach of finding fractional parts, what happens to the structure when children are then taught to use the algorithm in area situations?

Green found particular trouble in finding fractional parts of sets (e.g., 2/3 of 12). This topic was not identified as difficult in prior studies such as the ones by Morton (1924) or Brueckner (1928). However, in later Michigan studies (e.g., Muangnapoe, 1975), the set model for fractions was found to be difficult for children. Consequently, it is possible that the approaches to fractions and the operations on fractions that were used in the instructional materials of the 1960's may have created problems in finding fractional parts. There is a need to investigate whether finding fractional parts should be related more to whole number work on division than to fractions per se.

It was striking that the use of manipulative materials did not have the advantage either in achievement or attitudes often claimed by methods texts and curriculum designers. Evidently, it is much more complicated to relate a child's thought to his use of concrete materials and/or diagrams of the materials than is usually assumed. Later work by Payne, Greeno, Choate, Muangnapoe, Williams, and Ellerbruch (1974) and Muangnapoe (1975) attested to the need for carefully relating concrete objects to diagrams in teaching initial fractions concepts.

Green found much trouble with equivalent fractions, particularly with reducing to lowest terms, and Bidwell found similar difficulties with equivalent fractions. Consequently, because equivalent fractions is a topic needed for all operations, the next Michigan study was on equivalent fractions. Thus, a topic that had proved to be difficult to learn provided the direction and impetus for the subsequent study.

Three Sequences for Equivalent Fractions (Bohan, 1970)

Bohan designed three different instructional sequences for equivalent fractions. One used diagrams to develop equivalent fractions, one used a paper-folding technique, and a third introduced multiplication first and then the property of one. Bohan taught each sequence himself in two intact fifth grade classes (n = 171) in Huntsville, Texas. Each of the sequences was taught for six weeks. Details of the sequences are as follows:

Equivalent fractions (using diagrams), addition, and the multiplication (EAM). Diagrams of regions and number lines were the dominant models used to show that two fractions are equivalent. The key initial idea was essentially a measure idea; two fractions are equivalent if they show the same amount or the same measure. Then a "pattern" of multiplying numerator and denominator by a natural number was shown to yield the same results as those derived from diagrams. The generalization was arrived at inductively by looking at examples and observing patterns.

After three lessons on the meaning and language of fractions and six lessons on equivalent fractions, Bohan taught seven lessons and addition and subtraction of like and unlike fractions. Instruction was included on finding least common denominators, on building sets of equibalent fractions, and on using a factoring method. Then eight lessons were taught on multiplication.

Equivalent fractions (paper-folding), addition, and then multiplication (EPAM). This sequence was the same as EAM with only one exception. Paper-folding was used as a way to make a "logical" connection between diagrams that show equivalent fractions and the generalization of multiplying and dividing numerator and denominator by the same number.

Multiplication, equivalent fractions, and then addition (MEA). After three lessons on the meaning and language of fractions, Bohan taught six lessons on multiplication, eight lessons on equivalent fractions including one with stress on multiplication, and then seven lessons on addition and subtraction.

The order of topics was different in this sequence, but the major difference was the use of the property of one to develop the generalization for getting equivalent fractions.

Tests were administered on equivalent fractions, addition and subtraction, and multiplication at the end of instruction on the given topic. The retention test was given three weeks after the teaching ended. Arithmetic achievement was used as a covariate in adjusting means (see Table 3). Retention tests were not parallel forms of the posttest.

Bohan did an analysis of the various components of the three sequences that was similar to Bidwell's and Green's. The percentages of students attaining the sub-objectives in the sequences are shown in Tables 4-6.

There are many observations that can be made from Tables 3-6. Among them, the following seem to be of particular note:

1. EPAM is higher than EAM on C-3, the generalization of multiplying numerator and denominator by the same number to get higher terms.

2. Results are low on getting equivalent fractions to lower terms in all three sequences with only about one-half the pupils achieving criterion. Bohan conjectured that part of the difficulty may be lack of recall of division facts or the relationship between multiplication and division. Also, he recommended the development of a better model for getting lower terms.

3. Equivalent fractions scores on the posttest for MEA are relatively low. Bohan thought that the "property of one" approach did not promote understanding of equivalent fractions, but was a valuable mechanical aid.

4. There is a large drop for retention on multiplying a whole number and a fraction for all sequences.

5. Bohan noted that finding common denominators and using lists of sets of equivalent fractions were not sufficient to be able to add unlike fractions. He suggested a possible weakness in the addition sequences was the use of lists of equivalent fractions. He thought that the lists promoted an "addition" idea rather than a "multiplication" or "division" idea for generating equivalent fractions.

Table 3

ANCOVA
Post- and Retention Tests from Equivalent Fractions
Study: Arithmetic Achievement as a Covariate (Bohan, 1970)

Test	Posttest			Retention Test		
Treatment[a]	EAM	EPAM	MEA	EAM	EPAM	MEA
	(sixteen possible)			(thirty-six possible)		
Equivalent fractions	11.8(3.6)[a]	12.0(3.4)*	10.7(2.9)	18.3(6.5)	20.9(8.4)*	18.1(5.0)
	(twenty-one possible)			(twelve possible)		
Addition and subtraction	13.9(4.7)	14.1(4.5)	14.9(4.2)	7.8(3.4)	8.9(3.0)	8.4(2.6)
	(eighteen possible)			(fourteen possible)		
Multiplication	16.1(3.4)	16.6(1.9)	16.1(2.2)	8.8(4.1)	9.5(3.5)	8.1(4.0)
				(sixty-six possible)		
Retention total (includes three part scores plus four on fractions)	--	--	--	39.4(11.4)	42.6(13.0)*	38.6(8.8)
				(forty-six possible)		
Retention total (balanced for three topics)	--	--	--	30.0(10.4)	32.5(12.0)*	29.6(7.8)

[a] Standard deviation in parentheses
*p < .05

Table 4

Percent of Pupils Attaining Subconcepts of the EAM Sequence--Posttest and Delayed Test--(Bohan, 1970)

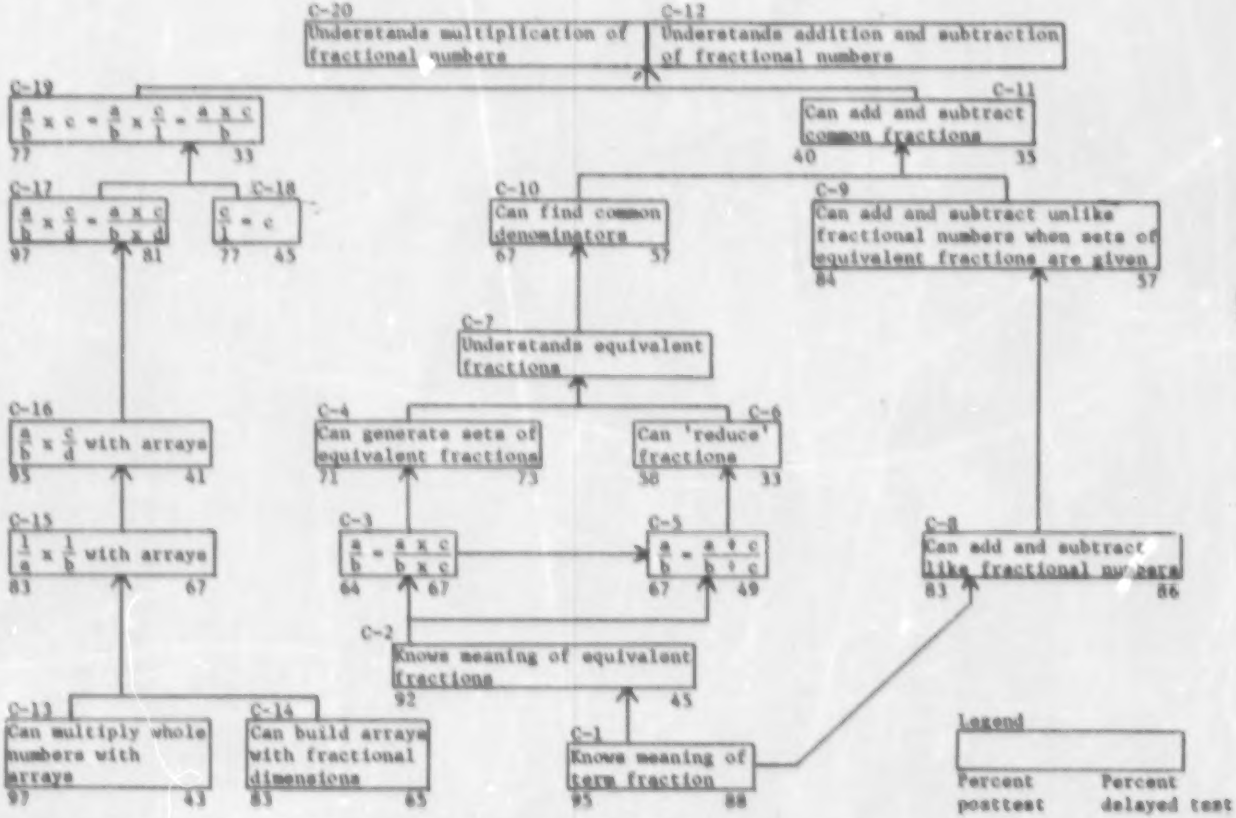

C-20
Understands multiplication of fractional numbers
C-12
Understands addition and subtraction of fractional numbers
C-19
$\frac{a}{b} \times c = \frac{a}{b} \times \frac{c}{1} = \frac{a \times c}{b}$
77
33
C-11
Can add and subtract common fractions
40
35
C-17
$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$
97
81
C-18
$\frac{c}{1} = c$
77
45
C-10
Can find common denominators
67
57
C-9
Can add and subtract unlike fractional numbers when sets of equivalent fractions are given
84
57
C-7
Understands equivalent fractions
C-16
$\frac{a}{b} \times \frac{c}{d}$ with arrays
95
41
C-4
Can generate sets of equivalent fractions
71
73
C-6
Can 'reduce' fractions
58
33
C-15
$\frac{1}{a} \times \frac{1}{b}$ with arrays
83
67
C-3
$\frac{a}{b} = \frac{a \times c}{b \times c}$
64
67
C-5
$\frac{a}{b} = \frac{a \div c}{b \div c}$
67
49
C-8
Can add and subtract like fractional numbers
83
86
C-2
Knows meaning of equivalent fractions
92
45
C-13
Can multiply whole numbers with arrays
97
43
C-14
Can build arrays with fractional dimensions
83
65
C-1
Knows meaning of term fraction
95
88
Legend
Percent posttest
Percent delayed test

Table 5

Percent of Pupils Attaining Subconcepts of the EPAM Sequence--Posttest and Delayed Test (Bohan, 1970)

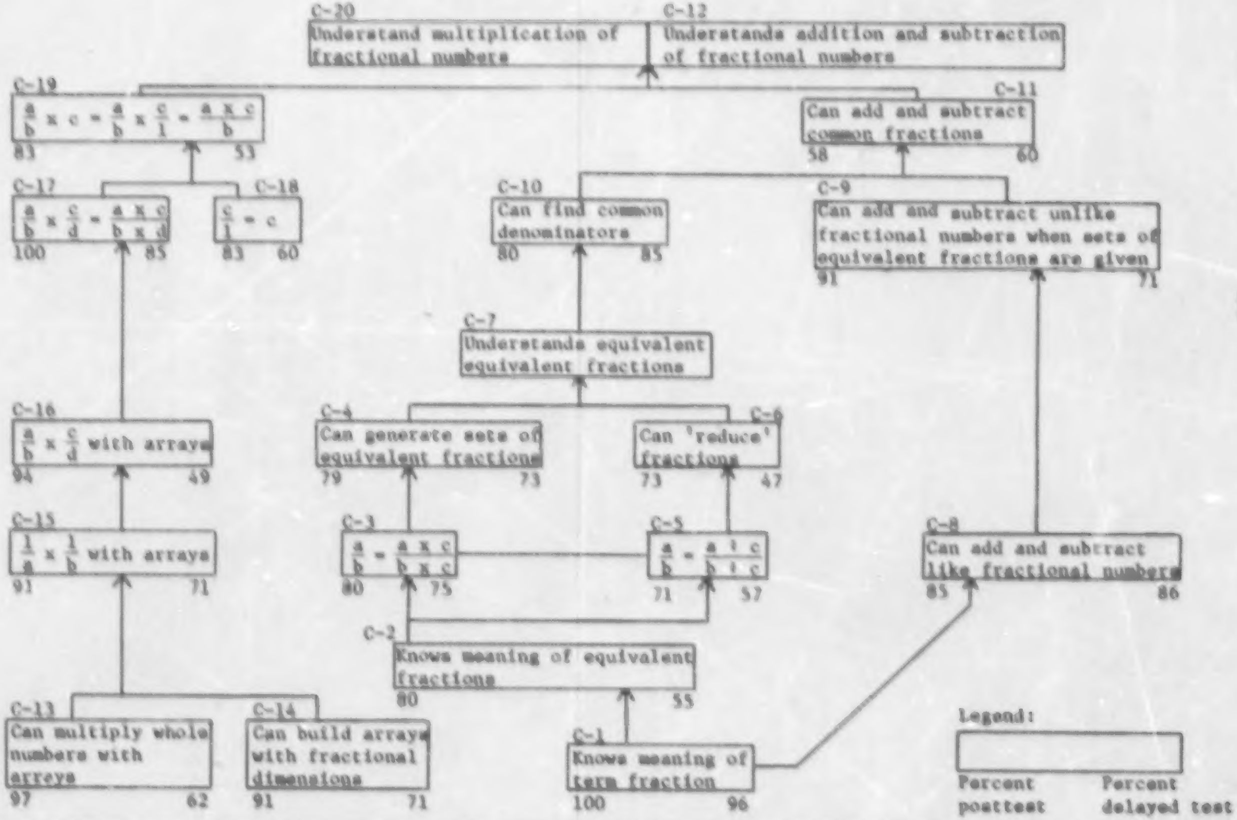
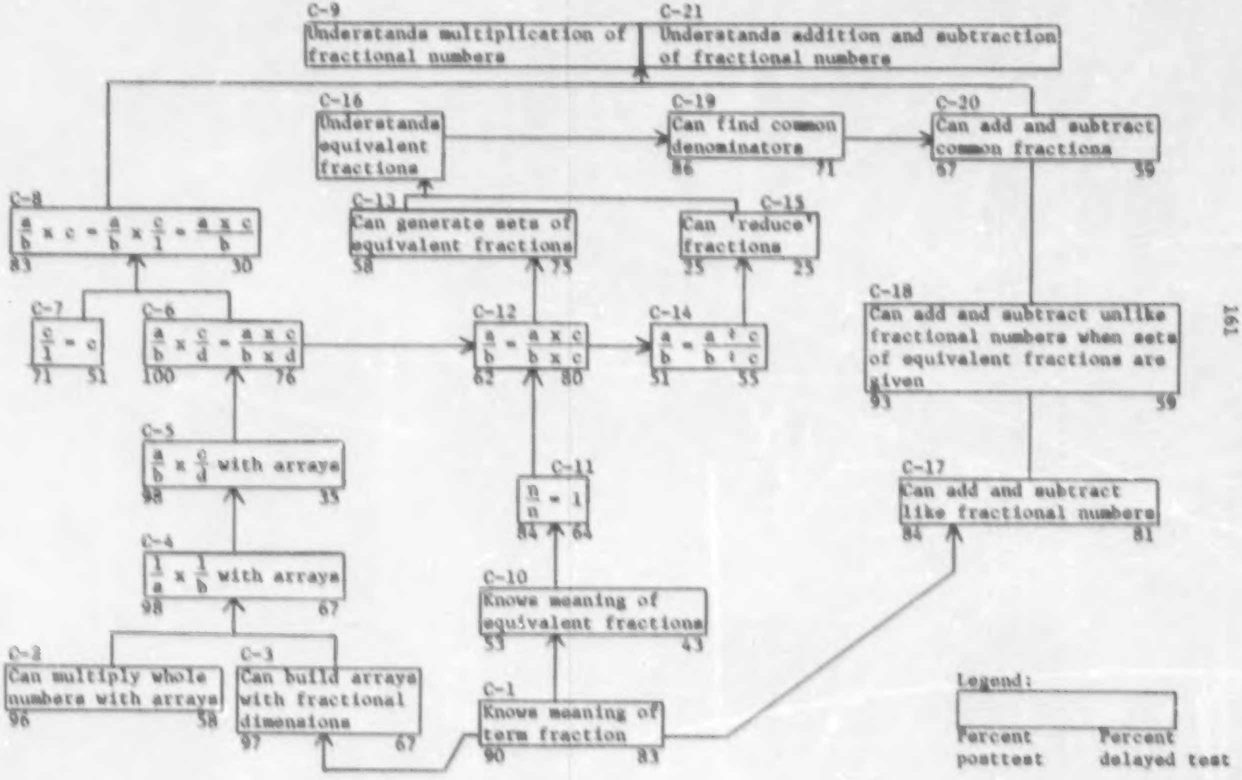

C-20
Understand multiplication of fractional numbers
C-12
Understands addition and subtraction of fractional numbers
C-19
a/b x c = a/b x c/1 = (a x c)/b
83 53
C-11
Can add and subtract common fractions
58 60
C-17
a/b x c/d = (a x c)/(b x d)
100 85
C-18
c/1 = c
83 60
C-10
Can find common denominators
80 85
C-9
Can add and subtract unlike fractional numbers when sets of equivalent fractions are given
91 71
C-7
Understands equivalent equivalent fractions
C-16
a/b x c/d with arrays
94 49
C-4
Can generate sets of equivalent fractions
79 73
C-6
Can 'reduce' fractions
73 47
C-15
1/a x 1/b with arrays
91 71
C-3
a/b = (a x c)/(b x c)
80 75
C-5
a/b = (a ÷ c)/(b ÷ c)
71 57
C-8
Can add and subtract like fractional numbers
85 86
C-2
Knows meaning of equivalent fractions
80 55
C-13
Can multiply whole numbers with arrays
97 62
C-14
Can build arrays with fractional dimensions
91 71
C-1
Knows meaning of term fraction
100 96
Legend:
Percent posttest
Percent delayed test

Table 6

Percent of Pupils Attaining Subconcepts of the MEA Sequence--Posttest and Delayed Test (Bohan, 1970)


C-9
Understands multiplication of fractional numbers
C-21
Understands addition and subtraction of fractional numbers
C-16
Understands equivalent fractions
C-19
Can find common denominators
86 71
C-20
Can add and subtract common fractions
67 59
C-8
a/b x c = a/b x c/1 = (a x c)/b
83 30
C-13
Can generate sets of equivalent fractions
58 75
C-15
Can 'reduce' fractions
25 25
C-18
Can add and subtract unlike fractional numbers when sets of equivalent fractions are given
93 59
C-7
c/1 = c
71 51
C-6
a/b x c/d = (a x c)/(b x d)
100 76
C-12
a/b = (a x c)/(b x c)
62 80
C-14
a/b = (a ÷ c)/(b ÷ c)
51 55
C-5
a/b x c/d with arrays
98 35
C-11
n/n = 1
84 64
C-17
Can add and subtract like fractional numbers
84 81
C-4
1/a x 1/b with arrays
98 67
C-10
Knows meaning of equivalent fractions
53 43
C-2
Can multiply whole numbers with arrays
96 58
C-3
Can build arrays with fractional dimensions
97 67
C-1
Knows meaning of term fraction
90 83
Legend:
Percent posttest
Percent delayed test

Bohan recommended the EPAM sequence (a) because of its demonstrated superiority, particularly in comparison with MEA, on equivalent fractions and (b) because of superiority in attitude measures over EAM and MEA. No method was superior for getting equivalent fractions to lower terms. The comparable achievement on multiplication shows that the time of introduction is not crucial. For higher achievement, Bohan suggested trying a sequence such as EPAM, but doing more during the multiplication sequence using the property of one to develop equivalent fractions. Of particular note is his observation that paper-folding helped make a logical connection between the concrete model and the generalization for getting higher terms.

Related research. While Bohan seemed to have devised a relatively effective sequence for getting higher terms, he was not successful with reducing (i.e., getting fractions to lower terms). Other researchers had also found trouble with equivalent fractions. Some did not always specify whether the problem was higher or lower terms. However, reducing has been identified as an area of greatest difficulty by most researchers.

In the Michigan studies, Bidwell and Green found both higher and lower terms difficult. Morton (1924) identified failure to reduce to be a major source of errors with fractions. Hinkleman (1956) found that the principle of multiplying/dividing the numerator and denominator by the same number was among the principles least frequently understood by fifth and sixth graders. Becker (1940) found that about one-third of the errors made by sixth grade students in addition of fractional numbers were attributable to equivalent fractions. Anderson (1969), while analyzing errors made by 599 fifth grade pupils on addition and subtraction of fractions, found that the two categories with highest percentages of errors related to equivalent fractions. The most frequent error Anderson found was renaming fractions in lowest terms and next was determining the numerator after the denominator of an equivalent fraction had been determined.

Of particular relevance to Bohan's study was the finding by Anderson that two methods were equally effective for finding equivalent fractions: (a) setting up rows of equivalent fractions or (b) factoring to find the lowest common denominator. Bat-haee (1969), however, found that the factoring method was superior to a "textbook" approach that used an inspection method involving equivalent fractions.

Steffe and Parr (1968) found little correlation between the ability of fourth, fifth and sixth grade students to solve equations such as $6/15 = n/5$ and their ability to handle proportional situations based on verbal or pictorial data. They suggested that this result indicated a lack of understanding of equivalent fractions.

Comment. The most significant conclusion in Boahn's study was the demonstrated superiority of the paper-folding technique for generating equivalent fractions in higher terms. It seems fair to conjecture that the folding itself provided a logical connection between observing two regions with the same measure and the generalization of multiplying numerator and denominator by the same number. Least effective was the sequence where pupils merely observed a pattern and inductively made the generalization.

Bidwell found similar trouble with his common denominator sequence; the sequence that was least logical of his three. Green, in retrospect, found greatest trouble with the Finding Fractional Parts approach. Consequently, it could be that the logical bridge is weak from finding fractional parts to the generalization of multiplying numerator and denominator by the same number.

Taking all three studies together, Bidwell, Green and Bohan, it appears algorithms seemed to be learned best when they were logical, and retention seemed to be best when there was a heavy visual component to instruction.

Of secondary importance from Bohan's study, but certainly worth noting, was the demonstration that instruction on operations on fractions can begin with multiplication. All textbooks and most methods books have suggested that addition and subtraction come first. Those few methods books that have suggested that multiplication come first have done so because multiplication could then be used with the property of one to generate equivalent fractions, however, this has not been researched. While using the property of one seemed not as effective as the paper-folding for generating equivalent fractions, in all other respects, introducing multiplication first produced results just as good.

On retention, the percentages of pupils in the three sequences that met criterion on reducing fractions was 33, 47 and 25--percentages clearly below any mastery level. The corresponding percentages for getting fractions to higher terms were 73, 73, and 75. The results on higher terms are very close to what one might consider mastery learning. Thus, mastery of equivalent fractions in higher terms by fifth grade pupils seems feasible, but lower terms is not feasible using any of Bohan's sequences. Clearly, a better instructional sequence is needed for teaching reduction of fractions.

Ratio and Region Model for Equivalent Fractions (Coburn, 1973)

Sparked by performance results and by his own interest in ratio as a topic, Coburn compared a ratio model and a region model for generating equivalent fractions with fourth grade children. He also examined the effect of the two models on addition and subtraction of fractions.

In ten intact fourth grade classes (n = 254), five teachers taught one sequence and five taught the other sequence. The study was done in Pontiac, Michigan. Posttest measures included the various components of equivalent fractions as well as addition and subtraction of fractions. A retention test was given two and one-half weeks after the posttest.

RATIO. Four Weeks: Generalization for Higher and Lower Terms

Ratios of disjoint sets were used to develop generalizations for multiplying/dividing numerator and denominator by the same number. Equivalent fractions were defined as two fractions that show the same ratio. The fraction symbol was associated with ratio. The ratios were part-to-part comparisons.

One week: Equivalent fractions using region model. The fraction symbol was associated with shaded portions of a region, thus, using a part-to-whole comparison for fractions. Equivalent fractions were viewed as two fractions that named the same number. Bohan's paper-folding technique was used to generate equivalent fractions, using the region model. This is essentially a measure idea deemed necessary for the subsequent work with addition and subtraction.

One week: Addition and subtraction of fractions. The major concrete model used related addition and subtraction to the measurement ideas developed in the prior week.

REGION. Four Weeks: Generalization for Higher and Lower Terms

Equivalent fractions were developed using the region model described above.

One week: Addition and subtraction of fractions. This was the same as the treatment under RATIO.

One week: Use of ratio with equivalent fractions. The content during this week was the same as for the first six days in the RATIO treatment. Initially, there was a slight modification to make the transition from the part-to-whole comparison required in the region model to the part-to-part comparison for the ratio model.

Among Coburn's conclusions were the following:

1. RATIO and REGION were equally effective with the renaming generalization. The mean percentage correct for both higher and lower terms for both treatments was just slightly better then 50. He questioned equivalent fractions as a fourth grade topic.

2. Using ratio diagrams, REGION was significantly better than RATIO when a part-to-part or a part-to-whole comparison was required. RATIO students tended to select a part-to-whole response when a part-to-part comparison was required.

3. The region diagrams were more difficult for children to interpret than the ratio diagrams.

4. REGION was superior to RATIO on addition and subtraction with unlike fractions. With like fractions, achievement in the two treatments was comparable.

Percentages correct on addition and subtraction of unlike fractions were low: 32% for RATIO and 47% for REGION.

5. Achievement was higher for REGION on solving problems involving proportions, especially ratio word problems.

6. REGION was higher on retention items such as $a/b = ?/nb$.

7. REGION had higher attitudes.

8. Neither group of pupils seemed to utilize the generalization developed initially in the four-week unit when developing their second interpretation. In the second encounter, pupils tended to rely on perceptual clues or counting rather than the generalization.

9. The correlation between adding using an equivalent fractions chart and addition shown in symbol form was .758 for REGION, higher than correlations between addition and renaming generalizations or addition of like fractions.

Coburn recommended beginning with the region model because it provides a smoother connection with addition and subtraction of unlike fractions, better maintenance of $a/b = ?/nb$, and a smoother transition to a part-to-part ratio comparison.

Coburn recommended a delay in verbalizing the rule for equivalent fractions until the need arose in addition with unlike fractions. He suggested the continual use of concrete models through addition of unlike fractions. He pointed out a need for substantially greater attention to equivalent fractions in instructional materials.

Coburn also pointed to a need for further work in specifying the components underlying fractional numbers themselves--forms of representation, relating symbols, verbal expressions, and the various translations.

Related Research. Coburn's results on equivalent fractions in grade four were comparable to those obtained in the National Longitudinal Study of Mathematical Abilities (NLSMA) done by the School Mathematics Study Group (Wilson, Cahen, & Begle, 1968) at the end of grade five, end of grade six, and beginning of grade seven. In the NLSMA study, the mean correct was 56% at the end of grade five on equivalent fraction items when one denominator was a multiple of the other (Wilson, et al., p. 27). At the end of grade six, on items such as 1/3 = 16/?, the mean correct was 51% (Wilson et al., p. 39). At the beginning of grade seven, items requiring interpretation of region diagrams for equivalent fractions showed mean correct of 44% (Wilson et al., p. 58). The NLSMA involved a very large sample of students, giving even more creditability to the poor achievement on equivalent fractions.

Steffe and Parr (1968) did a survey of fourth, fifth and sixth grade pupils to determine their ability to solve specific problems involving equivalant fractions. All items involved solving for the missing number in a proportion statement in which the corresponding terms were multiples of whole numbers. Some of the items were symbolic and some were pictorial. Of the pictorial items, half involved a ratio interpretation and half involved a region interpretation. Ratio items involved comparison of disjoint sets, while fraction items showed shaded parts of region diagrams. Steffe and Parr found that ratio diagrams were significantly easier to interpret than region interpretations, and they conjectured that the ratio interpretation would be easier to teach children than the region (measure) interpretation (Steffe & Parr, p. 23).

Coburn did not find the ratio interpretation easier. He conjectured that the reasons for the differences observed by Steffe and Parr were the more complex language used in testing with the region diagrams and the greater complication in the drawings of regions to interpret equivalent fractions (Coburn, p. 56).

Of particular interest in subsequent Michigan studies was the observation by Steffe and Parr:

> Much more care must be taken in the fifth and sixth grades to develop a sequence of lessons which are designed to enhance children's ability to represent visual data mathematically in the case of ratio or fractions, indeed if that ability can be enhanced. (p. 26)

Comments. The most significant finding in Coburn's study was that pupils using the REGION approach initially did better on addition and subtraction with unlike fractions. Thus, it seems evident that in choosing an initial model to be given main emphasis, one must look ahead to subsequent work. Comparability in achievement at the end of the development of the initial model, such as at the end of the initial development of REGION and RATIO, does not give one a free choice of either approach. Given equal achievement using the two models, the extent of use in tasks yet to come probably should determine the choice.

The biggest surprises in Coburn's work were: (a) the superior performance of the REGION group when part-to-part or part-to-whole comparisons were required and (b) the higher achievement of the REGION group on solving ratio word problems. Thus, it appears that ratio ideas are more readily accomodated to region ideas than conversely. Retention seemed to be better also for the REGION groups. These data and other evidence point to a more consistent and sound cognitive structure being built using REGION as the initial model.

In Coburn's sequence on addition and subtraction, he spent four days doing the operations using paper strips and fraction-bar diagrams. Then the generalization for equivalent fractions was used to do further work with addition. There was a very high correlation between success in adding/subtracting using the fraction-bar diagrams and success in adding/subtracting in symbolic form using the equivalent fraction generalization ($r = .758$ on posttest and $r = .838$ on retention test for REGION group). The correlations were not nearly as high between adding/subtracting in symbolic form with higher terms generalization ($r \approx .5$), lower terms generalization ($r \approx .4$), and addition subtraction with like denominators ($r \approx .4$). It seems likely that improvement on adding/subtracting with unlike denominators in symbolic form might come by spending more time initially with concrete materials, or else more time is needed to relate the symbolic form to the concrete materials.

At the conclusion of Coburn's study, there is added evidence on the difficulty with equivalent fractions, particularly in getting lowest terms. If one is to expect mastery learning of this topic, then more time is needed, and mastery probably should not be expected until the later grades.

At this point in the Michigan studies, there was strong evidence that performance on fraction operation algorithm was less than at mastery levels. Logical developments of algorithms seemed important. Concrete objects that seemed to fit well with steps in the algorithms appeared to help achievement, and this was more pronounced on retention tests. There is a question recurring in all the studies and particularly in the mind of this writer. The question is: What is the relationship between the concrete models, including the basic developmental work, and the rules for an algorithm by which an answer to a computational exercise is generated?

Initial Fraction Concepts, Symbols and Language (Galloway, 1975; Muangnapoe, 1975; Payne, et al., 1974; Williams, 1975)

In the period from 1973 to 1975, a series of related studies was conducted on learning initial fraction concepts and on algorithm learning. With an initial goal of having more pupils master the content being taught, more pilot work was done with small groups before using the instructional sequences with larger groups in more controlled experimental situations.

The central question under investigation was stated at the end of the previous section. All those invloved in the research project felt that any algorithmic work on operations with fractions must rest on sound knowledge of the initial fraction concepts. Furthermore, the investigators thought they knew how to achieve mastery on the initial concepts. Such was not the case, as was found in Pilot Study One.

Pilot study one. In the first pilot study, a team of seven investigators (Payne, Choate, Ellerbruch, Greeno, Muangnapoe, Galloway and Williams) designed an initial sequence on fractions, using as a guide the treatment found in most textbooks. The content was taught by investigators well prepared in mathematics and in mathematics education and experienced in teaching mathematics.

The initial pilot study was done using an entire fourth grade class. There was stress on the use of ordered pairs of natural numbers, written a/b, to name parts of regions, parts of segments, and parts of sets. While some paper-folding was done the first day, the major physical representation was done with diagrams. Included were mixed forms, fractions greater than one, comparing fractions and writing equivalent fractions using diagrams.

This pilot work showed that too much content had been presented with mastery achieved on almost none of it. Difficulties were observed with: (a) recognizing the need for equal size parts, (b) recognizing when parts were the same size in diagrams, (c) ignoring the fixed unit for fractions greater than one, (d) anything related to number lines, (e) reversing numerator and denominator, and (f) using the fraction symbol to express quantitative ideas.

Pilot study two. Because of the difficulty in assessing specific learning strengths and weaknesses in the whole classroom setting, the second pilot study used only five children, including one of high and one of low ability. One experimenter taught while the others observed. Revisions were made on a day-to-day basis, depending upon the outcomes of discussions. The discussions were time consuming because it was often difficult to get agreement on what the children were thinking.

In the seond pilot study, fractions were first introduced as a part of a unit (e.g., a sheet of paper or a region diagram) and later as part of a set. Number pairs were stressed, and language such as "2 of 4" or "2 out of 4" was used. Fractions greater than one and mixed forms came later. Much more time was given to the number line.

Much improvement was noted, but still there was not mastery of our objectives. Achievement was low on number lines, fractions greater than one, and equivalent fractions. There was much discussion about the trouble with using sets of objects as a model. An attempt was made to assess the pros and cons of including this model in the instructional sequence.

Pilot study three. While achievement was nearing mastery, more specific information was needed. It was decided to teach one pupil at a time. Muangnapoe, Choate, Ellerbruch, and Payne taught two pupils, one at a time for ten lessons. The second pilot study led to major changes in content taught in the third pilot study. Among the major features were: (a) the measurement idea was the overall theme for fractions, (b) concrete objects were used for a longer period of time before diagrams were used (The experimenters started to realize the perceptual difficulties children were having with diagrams.), (c) word names were taught before fraction symbols to help with the "reversal problem" (see Gunderson, 1957, for a description of the "reversal problem"), and (d) the set model was kept, but hesitantly because of the prior trouble it had introduced.

Using the names first eliminated the reversal problem completely. Difficulties persisted in identifying fractional parts of a set and with number lines. And, it was agreed that the set model was too difficult to include in the "final" sequence. The conjecture was that the set model interfered with the measurement idea.

Initial fraction sequence used by Muangnapoe. Muangnapoe (1975) studied the initial fraction sequence (IFS) very carefully, assessing with care the learning of all facets of the initial fraction work. He compared results obtained from four regular classroom teachers in grades three and four with those from his own instruction of 15 pupils in groups of three. A complete description of the sequence is in Payne et al. (1974)

The IFS spent four lessons using concrete objects. The first three lessons used sheets of paper and paper straws as units. Paper and straws were cut or folded and the oral fraction names "half, third, fourth, . . ." were taught as well as the written word names. The fourth lesson related the fraction symbol to the concrete representations. There were two lessons relating the fraction symbols to the word names for fractions and to diagrams of regions and number lines. Thus, there was a conscious plan to move from concrete objects to diagrams, with the diagrams always viewed as "pictures" of the concrete objects. Two final lessons were on mixed forms, including diagrams of regions and number lines. The final lesson was a review lesson. The posttest was given the next day. Retention tests were given about three weeks later.

Some of the data from Muangnapoe's study are reported in Table 7.

Table 7

ANOVA
Initial Fraction Sequence Pretest, Posttest,
Transfer and Retention, and Data on
Pupils Achieving Mastery Learning (Muangnapoe, 1975)

	Classroom Teacher (n = 91)			Muangnapoe (n = 15)		
Statistic	Range	Mean	SD	Range	Mean	SD
Pretest (thirty possible)	5-23	14.1	3.1	10-19	15.1	3.3
Posttest total (seventy possible)	34-70	60.2	9.0	41-70	63.3	8.1
Transfer (thirteen possible)	0-13	5.7	3.0	1-13	7.7**	3.1
Retention (sixty-nine possible)	35-69	58.7	8.9	47-69	64.3*	6.2
	Range			Mean		SD
Posttest (same as pretest)	12-30			25.7		4.8
	Number	Percent		Number	Percent	
Pupils with posttest scores ≥ 80%	64	70		13	87	
Pupils with posttest scores ≥ 90%	55	60		13	87	

*p < .05
**p < .01

There were some interesting results on transfer for pupils who achieved 90% or better on the posttest. For this group, the range on the transfer test was 2-13 with mean 7.35 (57%) and SD 2.68. All pupils who scored 8 or above on the transfer test came from this group. Thus, high achievement seems to be a necessary but not sufficient condition for transfer.

Items from NLSMA and from a standardized test were included on the posttest. On content taught in IFS, IFS achievement was substantially better.

Major difficulties persisted with: (a) identifying a unit from a diagram when more than one unit was shown, (b) realizing the need for equal-size parts of a unit (particularly obvious with circular regions using parallel cuts), (c) comparisons involving fraction symbols, (d) fractions greater than one, and (e) applying fractions to the number line.

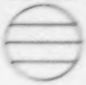
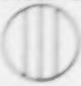
Comment. It took almost a full year to develop the initial fraction sequence to produce the unusually good achievement results reported in Muangnapoe's study. It is important to note, with these high results, that the set model had been dropped from the sequence because it seemed to interfere with the measurement ideas associated with regions and because achievement was low on the set model. Difficulties persisted with fractions associated with number lines, and errors on this topic contributed substantially to the lower scores of many pupils.

Subsequent to the Muangnapoe study, Greeno interviewed six pupils from the 90% or better achievement category: three who did very well on transfer and three who did not do well on transfer. The goal was to ascertain reasons for better transfer. After the interview, the conjecture was that those higher in transfer had sounder and more thorough knowledge than the others. This was added evidence that mastery of the initial fraction work is crucial. The major conclusion from Muangnapoe was that pupils in grades three and four can be taught so that they learn the subject well.

Initial fraction concepts in grades two and four and remediation in grade six (Williams, 1975). Williams made some modifications and used the sequence reported in Muangnapoe. Williams, herself, taught an intact class of second grade pupils, an intact class of fourth grade pupils, and ten remedial sixth grade pupils. In her action research project, she reported with some detail the reactions of pupils, noting with care the strengths and weaknesses in the pupils' learning. The pupils were in an inner-city school in Detroit in which she was an Assistant Principal.

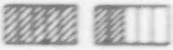
In grade two, she spent two weeks on instruction. Mean achievement on fraction concepts, word names and symbols using concrete materials was about 70% (p. 25). When Williams helped children make the transition from concrete objects to diagrams, she found difficulties emerging. Children had considerable trouble when rectangular, linear, and circular units were drawn on the same sheet. The number lines were especially difficult, both in identification and in construction of fractional parts.

At the end of instruction in the second grade, Williams (1975, pp. 29-30) summarized the students' progress as follows:

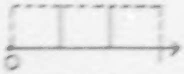
1. Most pupils were able to identify equal-size parts in non-circular units. However, some still considered  or  to be equally divided.

2. Improvement was noted in ability to locate and name diagrams representing one unit or less, except in relation to line segments.

3. Major difficulties persisting involved
   (a) construction of equal-size parts
   (b) conception of more than one unit as a unique quantity rather than as parts of a large unit; e.g.,

    was labeled as 7/10 rather than 7/5.

   (c) performing the same operations on line segments as on two-dimensional figures. One child converted the line segment into a rectangle; then proceeded to divide the rectangle to show the fraction named.



She did not think that instruction was as successful in grade four, probably because of irregular attendance of the pupils. Williams (1975, pp. 34-35) summarized the results as follows:

1. There was noticeable improvement in ability to recognize and name parts of units, except line segments.

2. Difficulty with fractions greater than one, and the use of mixed forms persisted.

3. Skill in accurately constructing representations of fractions revealed little progress. Units were hastily and unevenly divided, especially in diagrams. Folding was fairly accurate if edges could be brought together first, as with halves, fourths, and eighths. The narrow strips used as dividers were helpful in constructing thirds, fifths, and sixths. The relationship among halves, fourths, and eighths was recalled and used more readily than that among thirds, sixths, and ninths.

4. Although the entire sequence was covered, it appears to have been paced too rapidly for this fourth-grade group, especially under the adverse conditions of time and weather.

In grade six, Williams attempted to remediate ten especially low achieving pupils. She had only moderate success. Among comments Williams (1975) made were these:

> Posttest results indicated gains in ability to draw diagrams of one or more units and a specific part of congruent unit (other than line segments), when given a word name or fraction symbol; e.g., four thirds, 4/3. . . .
>
> The experiment does suggest that remediation may cause confusion for the child, in trying to sort out old and new concepts and determine which are more valid. It also suggests that, given a series of simple related tasks, some children are able to utilize past experiences and gain new insight into concepts previously misunderstood. (p. 42)
>
> The students appeared to understand the concepts quite well, especially in oral work; yet, worksheets and tests did not confirm this. In fact, test scores declined in some areas. (p. 47)

Williams suggested shorter written assignments, better geared to the slow learners' attention span.

Williams taped comments from her remedial students. Among the observations possible, Williams (1975, p. 48) alerted the reader to become aware of:

a. a child's difficulty in verbalizing ideas which he apparently understands;

b. a teacher's search for appropriate vocabulary to elicit from the child the rationale on which a choice was made;

3. a difference in how the child and the teacher perceive the same figure(s);

4. a child's fear of failure, or negative self-image;

5. the constant need for reinforcement; and

6. problems which symbolism presents—getting the words or numerals in exact order.

It seems clear from Williams' study that fractions related to concrete materials can be taught quite successfully by grade two. However, much remains to be done to develop successful remedial materials.

Initial fraction concepts, grades one through five, and decimals in grades three through five. Galloway (1975) examined the cross-sectional results of teaching the IPS used by Muangnapoe and Williams to pupils in grades one through five in a suburban school. Regular classroom teachers taught the unit, all at the same time of the year. The unit lasted 10 days. While the primary teachers, grades one through three, did not progress as far in the sequence, they did follow the same development. The primary teachers spent longer with concrete materials and diagrams, mixed forms were not introduced, and little was done with fractions greater than one. A three day decimal unit was taught to pupils in grades three through five following the IPS. Galloway, principal of the school in which the study was conducted, taught the decimal unit to the third grade class.

Among the findings by Galloway were these:

1. Students at all ages were able to correctly identify a unit, identify equal-size parts, and identify fractions less than one. At all age levels the concepts of "units" and "equal" were easy to learn. All levels could use oral fraction names, concrete models, and diagrams in fraction identification.

2. All age levels had difficulty with number lines. Primary students found writing and diagramming difficult.

3. Most students from the age of eight on can master the initial fraction concepts and symbols in a two-week period.

4. When comparing fraction achievement at various age levels, grade one students had a significantly lower achievement level than grades two or three. Grades two and three had achievement patterns which were very much alike in their posttests, but were significantly lower than grades four and five. Achievement for grades four and five were very much alike, with no differences at all on the retention test.

5. Decimals for tenths can be introduced and mastered by students in the third grade and above. All three levels achieved mastery in reading and writing decimals for tenths. There were no significant differences in achievement at the three levels. Pupils in grade three did have slightly more trouble reading a centimeter ruler drawing to tenths than did pupils in grades four and five.

Galloway recommended more attention to fractions in the primary grades, because of the good achievement and because of the high enthusiasm of the pupils in the early grades. She also recommended a longer period working with concrete materials, more time on teaching comparison of fractions, and more time on developing and interpreting number lines.

Comment. Results from Galloway and Williams show decisively that it is quite feasible to teach a substantial unit on fractions in the second grade and that high achievement is possible. Both Galloway and Williams recommend more work with concrete materials. Both recognized trouble with the number line and recommended substantially more time for the topic if it is included. Content expectations in grade one probably should be curtailed substantially from that in the IPS.

The success with decimals in grade three is probably attributable to the childrens' unusually sound knowledge of fractions. Their performance was astonishingly high both on decimals for tenths and on the centimeter ruler to tenths. They read and interpreted the centimeter rulers to tenths with great ease.

Algorithmic and Conceptual Development for Comparison of Fractions (Choate, 1975)

Choate's study reflects the initial concern of studying the way algorithms are presented. The major variable that he investigated was the time of presentation of the rule for comparing two fractions.

The general rule (R) used was (a) multiply each number of one fraction by the bottom number of the other fraction, and (b) since the fractions now have the same denominator, compare the fractions by comparing the numerators.

He used four treatments:

1. Rule Algorithm (RA)—Rule presented with no conceptual development.

2. Meaningful Algorithm (MA)—Each step of the rule was developed and illustrated with diagrams. The steps of the rule and the development went side by side.

3. Conceptual-Late Algorithm (CLA)--Diagrams were used extensively to compare fractions leading to a statement of the rule on the last day.

4. Conceptual-No Algorithm (CNA)--Same as CLA with deletion of the statement of the rule.

Two intact classes used each sequence. They were mixed fourth and fifth grade pupils, with the majority in the fourth grade. After teaching initial fraction work for two weeks, six lessons on comparisons were taught and a posttest given the seventh day. A retention test was given ten days later.

Achievement was comparable for the four treatments. The MA treatment tended to produce poorest results. Some subscores favored CNA, CLA and RA, but some favored MA. On transfer to addition and subtraction, CLA and CNA were significantly better than RA. Of some surprise was the superiority of RA on a transfer item solving proportion in number sentence form.

Comment. What seems clearest from Choate's work is that steps in a rule developed side by side with visual diagrams is the least effective way to teach an algorithm. While this result seems to contradict prevailing practice, it does make sense that there may be difficulty in teaching two seemingly different things at the same time.

The relatively good performance of the Rule Algorithm Group must be viewed with some caution. The pupils did have the IFS first and were well prepared on this initial fraction knowledge. Furthermore, some diagrams were used in the first day of instruction. Still, the overall performance of giving pupils only the rule is a bit of a mystery and seems to contradict what most mathematics educators might expect.

Ellerbruch is presently doing a study similar to Choate's but using the algorithm for addition and subtraction of fractions. (Results were not available when this was written, but the study has been completed since and is listed in the references.)

Directions for Research on Fractions

Two other documents seem important in assessing directions for further research on fractions. One is from cognitive psychology with emphasis on information processing. The other is a report on seventh grade pupils' thinking patterns in computation.

Greeno. For the past two years, James Greeno, a cognitive psychologist in the Psychology Department at Michigan, has been involved in discussions of the fraction research at Michigan. He participated at times in teaching and developing the initial fraction sequence. He gave freely of his time and his ideas in the construction of the sequences and in designing the lessons. He devised protocols and interviewed students at various junctures in the research in an effort to learn more about the cognitive structures the pupils were acquiring. And, he helped all of the investigators to deepen and to sharpen their knowledge of cognitive psychology.

Greeno has just written a chapter for a forthcoming yearbook (in press) that provides a highly useful psychological framework for the fraction research at Michigan. It is used now in retrospect because it has just appeared. It will be even more useful in future research. Some of the main points from his chapter are summarized here. Greeno should be held responsible for neither this summary nor these interpretations.

Perhaps the most important idea from Greeno (in press) for the fraction work is that quantitative concepts are procedures for working on spatially represented magnitudes. He gives a flow chart of the procedures used for the initial fraction concepts. The flow chart includes "images," and such geometric words as "regions" and "congruence." He views this as a spatial network. He cites evidence that comparions such as "Tom is taller than Bill" are done by using a representation with the properties of spatial ordering. After other evidence, he concludes that the concept of fractional quantity should be taught as a set of procedures for manipulating spatial magnitude. He sees spatial representation as an important component of problem solving.

He sees understanding of concepts and skills in computation as at least partially separable components of knowledge. He describes the process of getting equivalent fractions using regions and diagrams as a spatial processing. He sees the algorithm of multiplying/dividing the numerator and denominator of a fraction by the same number as being a different cognitive structure, a different concept of equivalent fractions.

Using examples from psychophysics, Greeno points to a relationship between some spatial processing routines and a network of primarily verbal concepts. He sees problem solving involving quantitative concepts as using a person's general mechanisms for language comprehension and concepts retrieved from semantic memory. This writer is uncertain of Greeno's view of the relation between spatial representation and verbal rules for computation that may or should exist in one's semantic memory.

Greeno's work is important because it provides a different and potentially more useful framework for describing the mental processes involved in mathematics learning. His spatial processing of information is useful in interpreting concrete materials and diagrams. His view and work on verbal rules should be helpful in understanding algorithmic learning.

Lankford. Lankford (1972, 1974) has reported in careful detail the thinking patterns revealed by 176 seventh grade pupils as they thought aloud in computing 13 whole number and 15 fraction exercises and in answering eight comparison questions (e.g., "Which is larger, 2/3 x 5 or 1 x 5?"). Incorrect algorithms were prevalent and varied for all the operations with fractions. On addition of fractions, 62 pupils added both the numerators and the denominators. With multiplication of mixed numbers by a whole number, often the whole numbers were multiplied and the fractions then merely affixed to the answer. A large number of pupils found a common denominator in doing both multiplication and division with fractions.

Lankford found that good computers (a) knew the "facts" and did not use primitive methods such as counting; (b) followed conventional algorithms rather consistently; (c) used paper and pencil more than would appear necessary; (d) did better on comparison exercises (all the comparison exercises involved fractions); and (e) sensed more often when an answer was wrong.

He found that poor computers (a) made extensive use of counting and often made errors with it; (b) had more trouble with fractions than with whole numbers; (c) devised what seemed as obvious procedures such as adding both the numerators and denominators; (d) often switched to a different algorithm to get an answer when they ran into trouble; (e) often supported by reason, even if faulty, what would appear to be careless errors; (f) had great difficulty with long division; and (g) frequently confused 0 and 1.

Lankford's careful documentation of the way children think and his astute observations on characteristics of good computers should help in designing more effective instructional sequences. This should help in research studies which aim at devising improved sequences for the various topics related to fractions.

With these additional documents in mind, suggestions are given for research on initial fraction concepts, equivalent fractions, addition and subtraction, and other topics.

Initial Fraction Concepts and Language

The major spatial representation for fractions in IPS was a region, including circular as well as rectangular ones. The number line was included. There were major difficulties with the number line model, with fewer students mastering this linear model than the area model.

When the sequence was taught, there seemed to be no trouble when teaching children to use straws as linear units. But there was trouble in using the linear idea with number lines. The trouble could have come because of the ordering required in making a number line. (There was trouble with comparison questions in the IFS.) Whatever the reason, there was trouble in making a transition from a straw concrete model to number line diagrams. Green (1969) had found a similar trouble in making a transition from concrete objects to diagrams in her study on multiplication of fractions.

Area of needed research: Number line model for fractions.

1. How does one make the transition from the straw model to a number line?

2. Does inclusion of "0" on the line cause trouble? If so, why?

3. Does extending the line always to the left of 0 and to the right, usually shown with arrowheads, facilitate or hinder the acquisition of this linear model?

4. More generally, how effective is the number line as an aid to the learner, and to what extent is it something extra to be learned?

5. How long does it take a child to "master" the number line model after learning the region model?

All investigators found that the set model for fractions was difficult to teach. After teaching the set model (e.g., 2/3 is 2 beans out of 3 or 2 of the 3), the region model with emphasis on measurement seemed to disintegrate.

The set model is very closely related to ratio; in fact, they may be the same thing. Since ratio is a topic most mathematics educators consider important, a way is needed to teach the initial set model and yet maintain the measurement model.

Area of needed research: Set model for fractions.

1. How does one relate a set model for fractions to the measurement model, assuming the measurement model is a part of the child's cognitive structure?

2. Is there an optimal time that should elapse after learning the measurement model before introducing the set model?

3. Does the usual way of introducing the set model for fractions require numerical and spatial networks that have not, in fact, been taught, (e.g., seeing that one-third of the set is circled (X X) X X X X)?

There is substantial evidence that achievement is low for interpreting a/b as a ÷ b (e.g., Bidwell, 1968). Junior high school and algebra teachers often cite this as evidence that kids know nothing about fractions. It does seem clear that this is an area of difficulty.

Area of needed research: Relating a ÷ b and a/b.

1. What is needed for pupils to make a meaningful connection between a ÷ b and a/b? To what extent is it an extension of the initial interpretation of fractions, and to what extent is it a new network to be learned? To what extent does the relation of a ÷ b to a/b involve the spatial network for initial fraction work?

2. At what age and with what knowledge background should the relationship between a ÷ b and a/b be taught?

Equivalent Fractions

There is evidence that relating a spatial representation and a verbal algorithm helps in learning equivalent fractions. The "action" model used in paper-folding seemed to produce the best results for Bohan.

Regardless of the approach, however, achievement results are low. Fourth graders in Coburn's study got only about 50% correct using either a ratio or a region approach. Green, Bohan, and other researchers still report poor overall results.

The better achievement on addition/subtraction produced when Coburn used a region model suggests that a ratio-proportion treatment for the initial work with equivalent fractions be delayed. This recommendation is consistent also with the difficulty in use of the set model in the IPS.

Area of needed research: Equivalent fractions.

1. In what ways should a spatial representation of equivalent fractions (regions and perhaps line segments) be related to the verbal rules for producing equivalent fractions?

Coburn's high correlation between spatial processing and similar problems using only fraction symbols is worth examining. Could it be that both the spatial model for equivalent fractions and verbal rules develop over a long period of time and that the development of verbal rules be viewed the same way? Or, should the spatial model and verbal rules approaches be viewed as completely different instructional strategies?

2. How much time is needed to teach equivalent fractions so that students master the topic? Is there an optimal age for teaching equivalent fractions?

3. Assume that equivalent fractions has been taught with heavy reliance on spatial models measuring regions and segments. Should the ratio model be taught separately, or should it be related to the measurement notion of equivalent fractions?

4. Why is there so much more difficulty with lower terms than with higher terms? Should they be taught at different times in the curriculum?

Area of needed research: Addition and subtraction of fractions.

In Bidwell's study on division of fractions, the two most successful algorithms, Inverse Operation and Complex Fraction, used spatial models very sparingly. On both models, there was a loss on retention after three weeks of 25-35%. Green's four approaches to multiplication did use spatial models, and her loss in retention on all treatments was only 10-15%.

This writer is convinced that there should be some spatial models, probably because they give better structures and also better retrieval from semantic memory. The way the rules and spatial models can and should be related is still a major question. This question arises as one views all the computational work with fractions.

When Ellerbruch and Payne were teaching addition/subtraction of fractions in grade four, they found that they could get almost 100% mastery of addition and subtraction of like fractions after mastery of the initial fraction concepts. Ellerbruch (1975) has further information (which was unavailable at the time this paper was written) on the relative effects of different placements of the verbal rules in his study on addition/subtraction where the spatial model is followed by the verbal rules, and conversely. Choate's results on comparison of fractions is convincing us that both verbal rules and spatial representation cannot be taught at the same time in parallel fashion.

1. Should addition and subtraction of fractions with unlike denominators be delayed until mastery of equivalent fractions?

2. How should the verbal and spatial representations for the addition/subtraction algorithm be related?

3. With relatively low achievement on addition/subtraction, should our instructional objectives be modified on the extent of content to be included in the curriculum and on the placment of the content in the curriculum?

4. With Lankford's results on pupils' frequent choice of an incorrect algorithm, one wonders about instructional strategies.

What is the effect of teaching an algorithm when many children do not "master" it? Does this make the algorithm more difficult to learn correctly later? Does the learning of an incorrect algorithm initially in effect double the time to learn it correctly (e.g., time to unlearn what is wrong and time to learn it correctly)?

Other Topics

Similar questions can be asked about the other computational topics with fractions. However, dependable information from one area may be easily generalizable to other areas and, in fact, may be generalizable to computation with whole numbers.

Overall, the Michigan studies show that it takes a very much longer period of time to teach any part of the fraction work than has generally been allowed in instructional materials or curriculum guides. Better and more reliable information is needed on the length of time it takes with various groups of children to teach fractions and operations on fractions.

In the Michigan studies, problem solving and applications have not been examined very carefully, although they have been included as dependent measures in many studies. There is much to be done in this area, especially with operations on fractions. There is strong evidence that finding fractional parts with sets is difficult, and it seems to be generally accepted that most pupils do not know when to use given operations on fractions. Some specific questions in this area are:

1. Why is it so difficult to learn how to find fractional parts (e.g., 2/3 of 12)?

2. Should finding fractional parts be related more strongly to fraction concepts or to whole number operations?

3. Does improvement in the development of spatial representations for fractions lead to better results on problem solving?

Summary

Overall, the Michigan studies began by comparing various approaches to an algorithm, always trying to understand better what is happening in the learner's mind. The initial comparative studies have evolved into studies aimed more directly at assessing the quality and durability of the cognitive structure the learners are building. With a national preoccupation for computation and with our own belief in quantitative thinking and problem solving, there is a need to address research more directly to the questions of how to produce each of these and what relation exists between them.

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Analysis and Synthesis of Research on Children's Ordinal and Cardinal Number Concepts*

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This paper is broadly concerned with the present state of knowledge about how and when children learn the concepts of ordinal and cardinal numbers. It is concerned, more particularly, with lawful age-related changes in these two concepts and with possible developmental asynchronies between them. The paper also deals, to a lesser degree, with the roles played by ordinal and cardinal number in the early growth of arithmetic skills. For the most part, the research that I shall review is descriptive. That is, the emphasis is on the findings and not on possible explanations. Although some explanatory hypotheses will be suggested, the formulation and testing of explanations is a process that is only just beginning. First, we must be sure of our facts. It is the purpose of this review to say what some of the facts are.

I do not propose to review the literature on children's prearithmetic numerical ideas exhaustively. An exhaustive review seems inadvisable for at least two reasons. First, my aim is to focus on developmental and functional relationships between ordinal, cardinal, and natural number. It happens that the bulk of the literature is irrelevant to this purpose. Virtually all studies ostensibly concerned with prearithmetic numerical skills focus primarily on cardinal number. Second, exhaustive reviews of studies concerned with both ordinal and cardinal number are already available (Brainerd, 1973a, 1975a, 1975b; Bryant, 1974; Flavell, 1970). For these reasons, I shall stress interpretation of the data herein. In particular, I shall stress an interpretative framework, the ordinal theory of number development, presented elsewhere (Brainerd, 1974, 1975a).

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Definitions and Terminological Conventions

Since our subject is a mathematical one, it may be helpful to begin with some general remarks about three concepts which occur repeatedly in the remainder of this paper, viz., ordinal number, cardinal number, and natural number. The remarks are in the nature of a logical characterization of each concept. I hasten to add, however, that these characterizations are far from what mathematicians would recognize as logically rigorous. My goal is simply to communicate most of the logical ideas on which ordinal, cardinal, and natural number rest. I am led to do this by the fact that most research on these concepts conducted by psychologists is only vaguely related to their logical characterizations. For example, in some studies, the ordinal number measure consists of learning to discriminate one position from another, and the cardinal number measure consists of learning to discriminate one manyness from another. Whatever the other merits of these studies may be, the concepts they measure are neither clear nor complete embodiments of ordinal, cardinal, and natural number. Therefore, it seems that research on children's number concepts could only benefit from some remarks on these concept's logical foundations. It is hoped that mathematical readers will forgive the imprecision of the remarks.

By "the system of ordinal numbers" we shall understand the unending sequence of numerals 1, 2, 3, . . ., where each numeral denotes one and only one term in some progression. By a "progression," we shall understand any collection of terms which (a) has a first term, (b) has no last term, (c) has no repeated terms, and (d) is such that every term can be reached from the first term in finitely-many steps. The ordinal number 1 is the symbol that we map with the "first" term of a progression, the ordinal number 2 is the symbol that we map with the "second" term of a progression, the ordinal number 3 is the symbol that we map with the "third" term of a progression, and so on.

The most common examples of progression are the arithmetic and geometric progressions of algebra. What these progressions have in common is an underlying transitive-asymmetrical relation. (The transitive-asymmetrical relation underlying any given progression is sometimes informally termed a generative law.) By a "relation," we shall understand any propositional function of the form R_{ab} where a and b are variables. In other words, a relation is any propositional function which contains two unknown terms. The values of a relation refers to any collection of terms which make the propositional function a true statement when they are used as interpretations of the variables. Statements of the form "a admires b," "a is shorter than b," and "a is the cube root of b" are relations. A relation is transitive if, for any three values x, y, and z of its variables, R_{xz} is true if R_{yz} and R_{xy} are both true. Thus, "a is shorter than b" is transitive, but "a admires b" is not: If Mary is shorter than Jim and Jim is shorter than John, then Mary is shorter than John; but if Mary admires Jim and Jim admires John, Mary may or may not admire John. A relation is asymmetrical if, for any two values x and y of its variables, R_{xy} is not true when R_{yx} is true and conversely. Because transitive-asymmetrical relations are what all progressions have in common and because ordinal numbers, as I wish to refer to them, are placeholders for the terms of progressions, the notion of transitive-asymmetrical relation is the basic logical idea underlying the concept of ordinal number and the concept of a progression is synonymous with the concept of transitive-asymmetrical relation. This simplifying convention will be made in spite of the fact that the latter concept is more general than the former two. For example, there are collections of values that satisfy the unknown terms of certain transitive-asymmetrical relations which are not progressions in the aforementioned sense. A collection of terms which satisfies the transitive-asymmetrical relation "a is a descendant of b," for example, is not necessarily a progression.

By the "system of cardinal numbers," we shall understand the unending sequence of numerals 1, 2, 3, . . ., where each numeral denotes a class of equivalent classes. By a "class," we shall understand any propositional function which contains one unknown term. By an "element" or a "member" of a class, we shall understand any interpretation (value, meaning, etc.) of the unknown term which makes the propositional function a true statement. Two classes R_a and R_b are said to be equivalent if for every member of R_a there corresponds one and only one member of R_b and conversely. If the correspondence between the respective members of the two classes is one-to-one, then the classes contain equally-many members. Two classes are said to be nonequivalent when the correspondence between their respective members is not one-to-one. The members of any class of equivalent classes are classes of the same manyness, where the relative manyness of two classes is determined by the type of correspondence between their respective members. Thus, while ordinal number is based on the idea of progression which in turn is based on the idea of transitive-asymmetrical relation, cardinal number is based on the idea of a class of equivalent classes which in turn is based on the idea of one-to-one correspondence between the respective members of two or more classes.

By the "system of natural numbers," we shall understand the unending sequence of numerals 1, 2, 3, . . ., where each numeral both denotes one and only one term in some progression and denotes a class of equivalent classes. Classical arithmetic took the natural numbers as its starting point. Today, we know that arithmetic can be founded on either the ordinal or cardinal meanings of natural numbers. The former fact was discovered during the latter half of the 19th century by Dedekind and Peano (e.g., see Russell, 1903, chaps. 24 & 29), and the latter fact was discovered somewhat later by Frege and Russell (e.g., see Quine, 1969). Although there seems to be no basis for choosing between the ordinal and cardinal meanings of natural numbers from a logical point of view, there is no denying that the ordinal meaning is historically more important in the development of mathematics. In Euclid's Elements, for example, the characterization of the natural numbers, though informal by modern standards, is purely ordinal. (See Definition 5 in Book V and the theory of proportions in Book VII.[1]) Similarly, during the second half of the 19th century, Cantor, Dedekind, and Peano were able to show that the numerical aspects of all classical mathematics--i.e., arithmetic, algebra, analysis, and geometry--depended only on the fact that natural numbers denote terms in progressions (see Russell, 1903, chap. 29). The first laws of classical arithmetic (associativity, commutativity, and distributivity), for example, were shown to depend on the fact that symbols which appear in them denote the terms of progressions (e.g., Dedekind, 1888). In view of their historical importance, 19th century mathematicians tended to view ordinal numbers as logically more fundamental than cardinal numbers. Of course, Russell (1903; Whitehead & Russell, 1910-1913) later showed that this was not the case, and (a) classical mathematics can be based on either ordinal or cardinal number and (b) the concept of ordinal number can be used to define the concept of cardinal number or the concept of cardinal number can be used to define the concept of ordinal number. Thus, either the ordinal or cardinal meaning of natural numbers will suffice for arithmetic and the rest of classical mathematics. Although there may be aesthetic and/or philosophical and/or psychological reasons for preferring one meaning over the other, there do not appear to be any purely logical grounds for choosing to base arithmetic on one rather than the other.

Ordinal Theory of Number Development

From the fact that ordinal and cardinal number are equally basic logically, it does not necessarily follow that they are equally basic psychologically. The theoretical framework which I should now like to summarize proposes that (a) ordinal number is psychologically more basic than a cardinal number, and (b) ordinal number plays a more important role in the early growth of arithmetic concepts and skills than cardinal number does. The theory was devised to explain the findings of a series of studies conducted in my own laboratories (Brainerd, 1973a, 1973b, 1974, 1975a; Brainerd & Fraser, 1975). These studies deal primarily with the developmental and functional relationships between ordinal, cardinal, and natural number. Since the theory reflects the present incomplete state of our knowledge about these relationships, it goes without saying that theory itself is incomplete. In fact, to be perfectly accurate, the theory is not so much a theory as a crude working hypothesis that requires constant revision as more data come in.

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[1] I am indebted to F. S. C. Northrop, Sterling Professor of Philosophy at Yale University, for drawing my attention to the fact that ancient Greek mathematicians in general and Euclid in particular used an ordinal definition of the integers.

To begin with, it is taken as axiomatic that any theory of number development, whatever else it may do, must try to explain the origins of arithmetic concepts in children's thinking. That is, it must try to explain how children come to recognize that statements such as "one plus one are two" and "four minus two are two" are true and statements such as "one plus one are seven" and "four minus two are three" are false. In short, a theory of number development must try to explain, however incompletely, how we first come to manipulate the natural numbers in ways consistent with the laws of arithmetic. To lose sight of this aim as, for example, Piaget's theory of number development does (see Brainerd, 1975a, chap. 6), is to create a theory of little general interest.

A theory of number development, like any other developmental theory, should be historical. That is, it should try to explain later acquisitions in terms of earlier ones. Historical explanations lead to two general kinds of empirical predictions--*ordinal* ones and *functional* ones. The former, as the name suggests, are concerned with the order in which two things (behaviors, concepts, etc.) are acquired. If one thing is a necessary condition for another, it follows that children should acquire that thing before they acquire the thing for which it is a necessary condition. Simple cross-sectional designs normally are used to investigate ordinal predictions. Evidence that one thing invariably precedes another only suggests that the earlier thing may be a necessary condition for the later thing. It does not prove the hypothesis. The confirmation of such a hypothesis requires the examination of functional predictions. Verification of such predictions is the sufficient condition for inferring that one thing is a necessary condition for another. Training experiments are used to study functional predictions. In such an experiment, we conclude that one thing is a necessary condition for another if training that thing produces correlated improvements (transfer) in the other thing.

The central assumption of the ordinal theory is that, at least in the beginning and probably later also, emerging arithmetic competence depends primarily on a prior grasp of ordinal number. Conversely, it is assumed that a prior understanding of cardinal number is not very closely connected with the emergence of arithmetic competence. As it now stands (as of early 1975), these proposals lead to three general types of predictions: (a) predictions about the development of ordinal number, (b) predictions about the development of cardinal number, and (c) predictions about the relationships between ordinal, cardinal, and natural number.

Development of Ordinal Number

According to the ordinal theory, the concept of ordinal number is ultimately rooted in a certain type of perceptual experience, viz., perceived ordinality. The notion of perceived ordinality is concerned with the perception of everyday transitive-asymmetrical relations (e.g., "taller than," "louder than," "heavier than") that underlie concrete empirical progressions. To be a source of perceived ordinality experiences, a given empirical progression must have three principal characteristics: first, it must have at least three terms (with two terms there is asymmetry but no transitivity); second, the pairwise relational differences between the various terms must all be large enough to be directly perceptible; third, the ordering of the relational differences between the various terms must be perfectly correlated with a common perceptual ordering of some sort (e.g., a spatial ordering, a temporal ordering). Three illustrative visual stimuli which produce perceived ordinality are shown in Figures 1A, 1D, and 1E. Each stimulus consists of more than three dots, and the underlying transitive-asymmetrical relation in virtue of which each group of dots forms a progression is "larger than." Note that, in line with the second and third of the preceding conditions, the pairwise size differences in each progression are highly perceptible, and the left-to-right ordering of each collection is perfectly correlated with the ordering of the size differences. With concrete progressions which meet all three of the preceding conditions, we believe that subjects are capable of directly perceiving the underlying transitive-asymmetrical relations at a very early age. Percepts of this sort are what the ordinal theory means by "perceived ordinality." Perceived ordinality may well be a natively-given percept in so far as vision and audition are concerned. For our purposes, however, it is sufficient that perceived ordinality appears several years before children evidence any understanding of arithmetic. A simple experiment suffices to show that even very young children are capable of perceived ordinality. Suppose we give a two-year-old a very brief exposure--say one or two seconds--to a stimulus which consists of Figure 1A. We then present a recognition stimulus which consists of Figures 1B, 1C, 1D, and 1E. The child is asked to select the figure which contains the same dots as the original stimulus. Figure 1D and Figure 1E will be most frequently chosen (Brainerd, 1975a, chap. 6). However, note that none of the dots in either of these progressions is the same as any of the dots in Figure 1A. The only thing that is the same about these three collections is the underlying transitive-asymmetrical relation. In contrast, Figures 1B and 1C are only rarely chosen. And yet, the dots in each figure are exactly the same as the dots in Figure 1A. But the underlying ordering of the dots in terms of their relative size is not apparent to direct perception. It is from perceptual experiences of the sort illustrated in Figures 1A, 1D, and 1E that the concept of ordinal number evolves--or so the ordinal theory supposes.

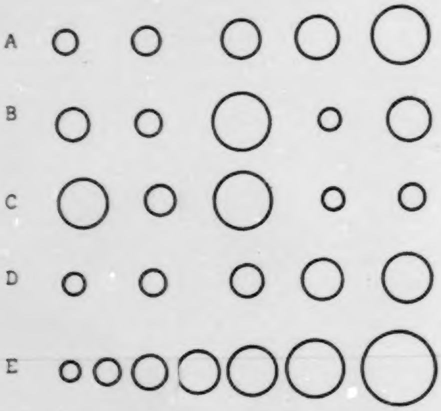


Figure 1. Some illustrative stimuli for studying perceived ordinality.

From the antecedent ability to perceive ordinality, internalized concepts of order develop during the preschool years. Our most recent evidence (Brainerd & Siegel, 1975; Siegel, 1974) indicates that the development of such concepts begins no later than age three, and, in all probability, it begins substantially earlier. To date, research on the early growth of internal ordering concepts has been confined to observing age-related improvements in such concepts. The process whereby these improvements take place is, as yet, poorly understood. There is reason to believe that the laws of discrimination learning are involved. There also is reason to believe that the linguistic and imagery skills which emerge during the preschool years are involved. However, it is impossible to be more specific than this at present. The emergence of internal ordering concepts during the preschool years is suggested by children's capacity to solve two general types of ordering problems. The first type of problem is what I call discriminative ordinality. As the name suggests, problems of this sort, unlike perceived ordinality, involve at least some learning. Discriminative ordinality problems involve stimuli which satisfy only the first two of the above conditions for perceived ordinality stimuli--i.e., they contain three or more terms, the relational differences between terms are all highly perceptible, and the terms' spatial or temporal order is uncorrelated with the ordering of their relational differences. Figure 1B and Figure 1C are stimuli of this sort, as are Figures 2A and 2B. Figure 2 may be used to illustrate a typical discriminative ordinality problem. The subject is first presented with a series of stimuli which each contain the dots shown in Figure 2A; the left-to-right ordering of the dots differs from one stimulus to another. The subject is reinforced for making the transitive-asymmetrical response "middle-size." When the subject has learned this response to some criterion, he or she is shifted to a new series of stimuli. Each stimulus in the new series contains the three dots shown in Figure 2B, and the left-to-right ordering of the dots is different for different stimuli. The subject is again reinforced for middle-size. If the subject did not learn the correct ordinal discrimination on the first series of trials, but instead, simply learned to respond to a certain absolute size, there should be pronounced negative transfer with the second set of stimuli. However, if the subject learned the correct ordinal discrimination on the first series of trials, there should be no negative transfer. The latter is what we find, at least down to and including age three. Some illustrative data from a recent experiment of this sort conducted by Linda Siegel and I appear in Figures 3 and 4. The curves in Figure 3 show the average numbers of middle-size responses made by three- to six-year-olds with a series of stimuli like Figure 2A during an initial block of 24 discrimination learning trials. The curves in Figure 4 show the average numbers of middle-size responses made by the same subjects with a series of stimuli like Figure 2B during a second block of 24 discrimination learning trials. Note that (a) there is no evidence of negative transfer from the first set of stimuli to the second, and (b) the performance of even three-year-olds was far above chance. Hence, it seems that by age three, and perhaps considerably earlier, there is good evidence of discriminative ordinality in North American children.

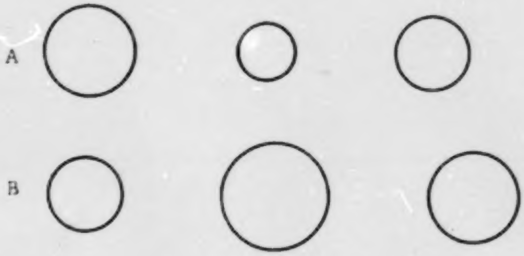


Figure 2. Some illustrative stimuli for studying discriminative ordinality.

The second group of problems whose solution seems to suggest the presence of internal ordering concepts is what I call *ordination*. Ordination problems are designed to be fairly precise embodiments of the logical definition of ordinal number (Brainerd, 1973a, 1975a, chap. 7). As discriminative ordinality problems were one step more difficult than perceived ordinality, so ordination problems are one step more difficult than discriminative ordinality. More particularly, ordination problems involve progressions which satisfy only the first condition for perceived ordinality stimuli--i.e., they contain three or more terms but the relational differences between terms are not highly perceptible and the spatial or temporal ordering of the terms is uncorrelated with the ordering of their relational differences. There are two general types of ordination problems, viz., *seriation* and *transitive inferences*. Concerning the former, a subject is given three or more objects that differ in terms of some common transitive-asymmetrical relation ("taller than," "longer than," "heavier than," etc.). The pairwise asymmetries between the various terms are small and not apparent to direct perception. The subject is instructed, first, to determine all the pairwise asymmetries by comparing the objects and, second, to put the objects in order from least to greatest or greatest to least. Two illustrative problems are shown in Figure 5. On the right in Figure 5, the subject is given three balls which look identical but which actually differ in relative weight. The subject's task is to determine the direction of the weight difference between each of the three possible pairs and then to arrange the three balls in order. At the left of Figure 5, a similar task is shown which involves three sticks that differ in length by tiny amounts. Transitive inference ordination problems resemble seriation ordination problems. The only difference is that subjects are not allowed to make all possible pairwise comparisons. Instead, they are required to deduce some pairwise asymmetries from others. To illustrate, suppose that the three balls shown at the top of Figure 5 weigh 50 grams, 100 grams, and 150 grams, respectively. In a typical transitive inference problem, the subject would compare the 50 gram ball with the 100 gram ball, compare the 100 gram ball with the 150 gram ball, and then be required to deduce the relationship between the 50 gram ball and the 150 gram ball. In the ordinal theory, the solution of transitive inference problems is viewed as the chief indicator of the concept of ordinal number. The fact that certain asymmetrical relations are transitive, the concept that this problem is supposed to measure, is called the *minimum ordinal proposition* (cf. Brainerd, 1973a, 1975a, chap. 7; Russell, 1903), and it is the fact that makes progressions logically possible.

Since ordination problems are more difficult then discriminative ordinality problems, it is assumed that they are solved somewhat later than discriminative ordinality problems. In particular, while the ordinal theory proposes that discriminative ordinality is a late-infancy or early-preschool acquisition, the theory proposes that ordination is

Successive Ordinal Number Discrimination

Block #1

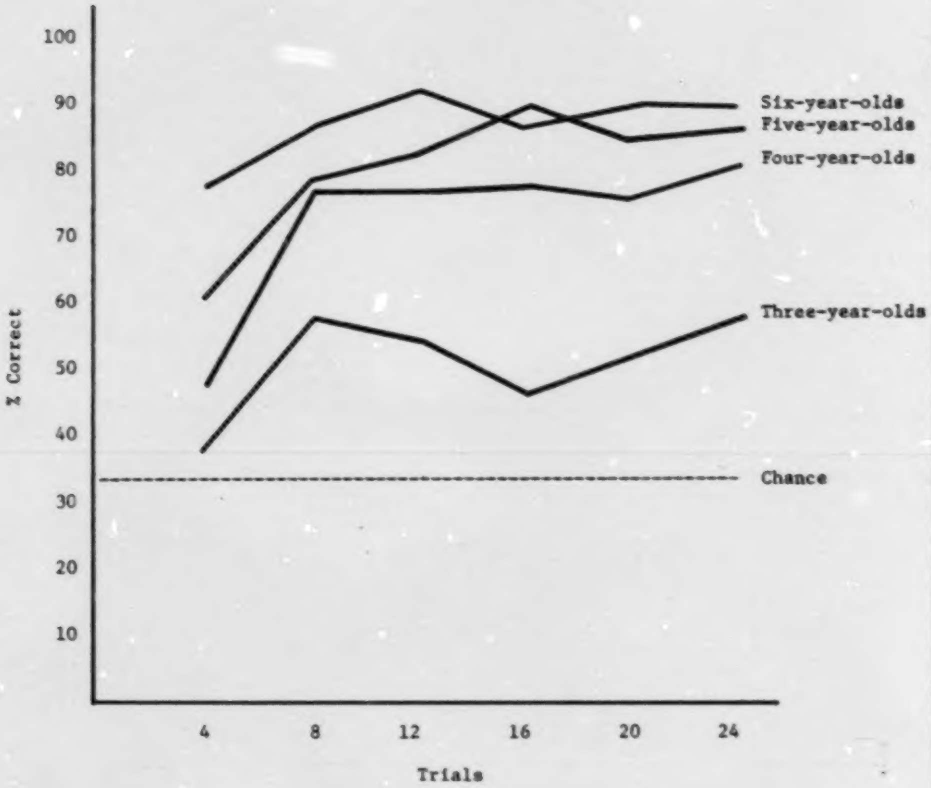


Figure 3. Average numbers of middle-size responses made by three- to six-year-olds on stimuli resembling Figure 2A. N = 35 per age level.

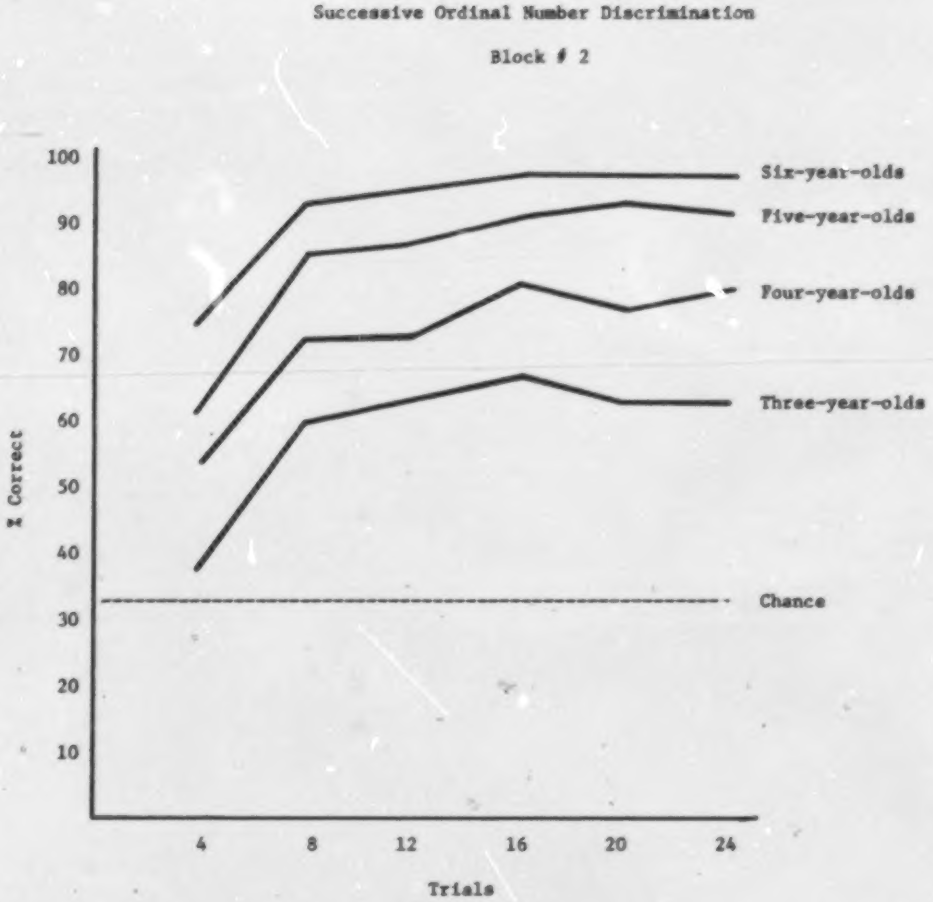


Figure 4. Average numbers of middle-size responses made by three- to six-year-olds on stimuli resembling Figure 2B after prior trials with stimuli resembling Figure 2A. N = 35 per age level.

a middle-to-late preschool acquisition. The ordinal theory also proposes that the overwhelming majority of children are capable of ordination by the time they enter kindergarten. Since ordination problems are cognitive counterparts of the logical definition of ordinal number, this latter proposal amounts to saying that most children already grasp ordinal number by the time they enter kindergarten.

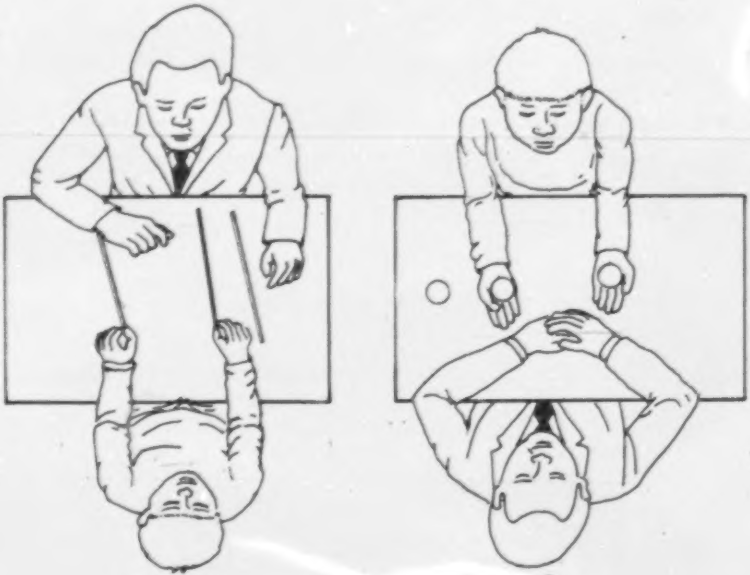


Figure 5. Two illustrative situations for assessing ordination.

Development of Cardinal Number

The ordinal theory proposes that the concept of cardinal number, like the concept of ordinal number, ultimately is rooted in a particular type of preconceptual perceptual experience. This category of percepts is variously termed numerousness (e.g., Underwood, 1966), numerosity (e.g., Nelson & Bartley, 1961), and, as I shall refer to it, perceived cardinality. Perceived cardinaltiy is concerned with the direct perception of manyness differences between pairs of collections. Given certain special conditions, we know that preschoolers are able to perceive manyness differences between pairs of classes. The special conditions are (a) the manyness difference must always be very large; (b) neither collection may contain more than a decade of terms; (c) at least one perceptual difference is correlated with the manyness difference. A stimulus which satisfies these conditions is shown in Figure 6. There are two and two-thirds times as many dots as triangles; the largest collection contains only an octet of dots; both the dots and the triangles occupy the same amount of space so that the relative manyness of the two collections is correlated with their relative density. With concrete classes which meet all of these criteria, we know that, by about age four, most children will be capable of directly perceiving the direction of the manyness difference. For example, if four-year-olds are given brief exposure--say, one or two seconds--to a stimulus like Figure 6 and are then asked, "Were there more dots or more triangles?", they will usually respond correctly (Taves, 1941). Since the brief exposure time precludes either counting the collections or establishing a correspondence between their respective elements, it follows that correct judgments are perceptual rather than conceptual in nature. The ordinal theory maintains that the human cardinal number concept evolves from perceptual experiences such as these.

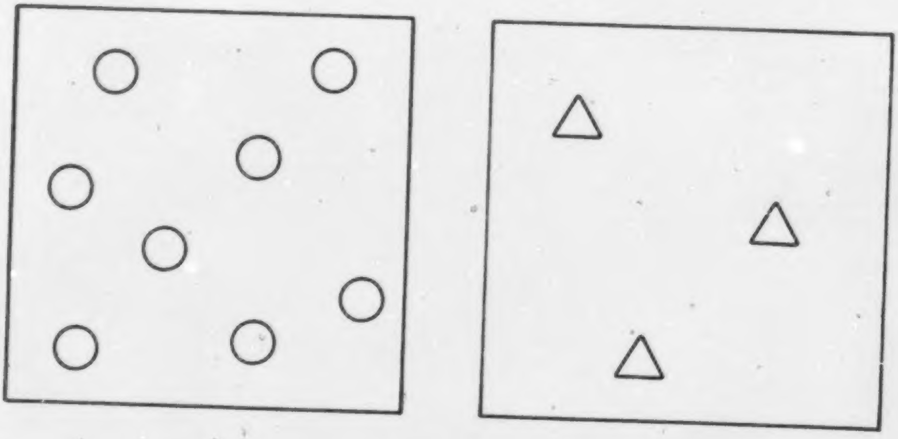


Figure 6. An illustrative stimulus for studying perceived cardinaltiy.

Internalized concepts of cardinality begin to evolve from perceived cardinality during the late-preschool and early-elementary school years. Recent findings (Brainerd & Siegel, 1975) suggest that this process begins during the fifth year of life in most children. As was the case for the development of internal ordering concepts, the currently available evidence does not allow us to be very precise about the underlying mechanisms that are responsible for the acquisition of internalized cardinality. As was also the case for the development of internal ordering concepts, the emergence of internal cardinal skills is suggested by children's solution of two general types of problems. The first of these, discriminative cardinality, is the cardinal counterpart of the aforementioned discriminative ordinality problems. Discriminative cardinality, like discriminative ordinality, is based on learning rather than perception. Explicitly, discriminative cardinality consists of learning any one of the first few cardinal numbers (i.e., a unit, a pair, a trio, ...). Discriminative cardinality stimuli satisfy the last two of the preceding three criteria for perceived cardinality but not the first. Discriminative cardinality involves stimuli consisting of three or more classes between which there are small manyness differences (a single term or a pair of terms). None of the classes contains more than a decade of terms, and a perceptual cue (e.g., density or length) is correlated with relative manyness. Figure 7 and Figure 8 are stimuli of this sort, and they may be used to illustrate a typical discriminative cardinality problem. The subject first is shown a series of stimuli like Figure 7. All the stimuli contain three classes consisting of a pair of dots, a trio of dots, and a quartet of dots, respectively. The left-to-right ordering of the classes is varied from one stimulus to another, but, since each class occupies the same amount of space, relative density is perfectly correlated with relative manyness. The subject is reinforced for selecting a particular manyness (e.g., he or she is reinforced for choosing "quartet"). After this response has been learned to some criterion, the subject is shifted to a new series of stimuli. Each stimulus in the new series contains the three classes shown in Figure 8. The ordering of the classes again varies from one stimulus to another, but relative density is perfectly correlated with relative manyness. The subject once again is reinforced for choosing "quartet." If the subject did not learn the correct manyness discrimination on the first series of trials but, instead, learned the transitive-asymmetrical discrimination "largest," we would expect clear negative transfer to the new series of stimuli. However, if the subject learned the correct manyness discrimination on the first series of trials, there should be no negative transfer. Prior to about age six, we find massive negative transfer. Some illustrative data from a recent experiment on this question conducted by Siegel and I are shown in Figures 9 and 10. Figure 9 shows the average numbers of correct quartet responses made by three- to six-year-olds on a series of stimuli like Figure 7 during an initial block of 24 discrimination trials. Note that although discriminative cardinality appears to be more difficult for each of the age groups than discriminative ordinality (compare Figure 3 with Figure 7), even the three-year-olds appear to be performing above the chance level. Figure 10 shows the average numbers of quartet responses made by the same subjects on a series of stimuli like Figure 8 during a second block of 24 discrimination trials. Note that (a) there is a massive negative transfer at all age levels and (b) only the six-year-olds appear to be performing above the chance level. This suggests, at least to Sigel and I, that discriminative cardinality emerges sometime between age five and age six and that, before this age, children learn a purely relational discrimination on discriminative cardinality problems.

The second and more crucial type of problem used to assess internal cardinal concepts is cardination. As ordination problems are designed to be precise embodiments of the logical definition of ordinal number, so cardination problems are designed to tap the components of the logical definition of cardinal number (Brainerd, 1973a, 1975a, chap. 7). That is, cardination problems are presumed to evaluate a given a child's understanding of the connection between type of correspondence and relative manyness. Cardination problems involve stimuli which are slightly more difficult than those just described for discriminative cardinality. Cardination problems involve sets of stimuli consisting of pairs of classes which, first, contain no more than a decade of terms, second, between which there is either no manyness difference or only a very small difference, and, third, no perceptual cue is perfectly correlated with relative manyness. A set of six such stimuli appears in Figure 11. Each of these stimuli consists of two parallel rows of dots; the upper row is always red and the lower is always blue. In conjunction with each stimulus, a given subject is asked to judge whether or not the reds are more numerous, less numerous, or equally numerous as the blues. Two perceptual cues, length and density, are available. However, neither is an infallible guide to relative manyness. If the subject depends on the length cue (i.e., longer = more numerous), then he or she will miss all the items. If the subject depends on the density cue (i.e., denser = more numerous), then he or she will miss at least two of the items. The only way, apart from counting the terms in each class, that the subject can always respond correctly is by establishing a term-by-term correspondence between the reds and blues. If we forbid counting, this leaves only correspondence. In the ordinal theory, the solution of sets of problems such as these is the principal datum on which the conclusion that a given subject possesses the concept of cardinal number is based.

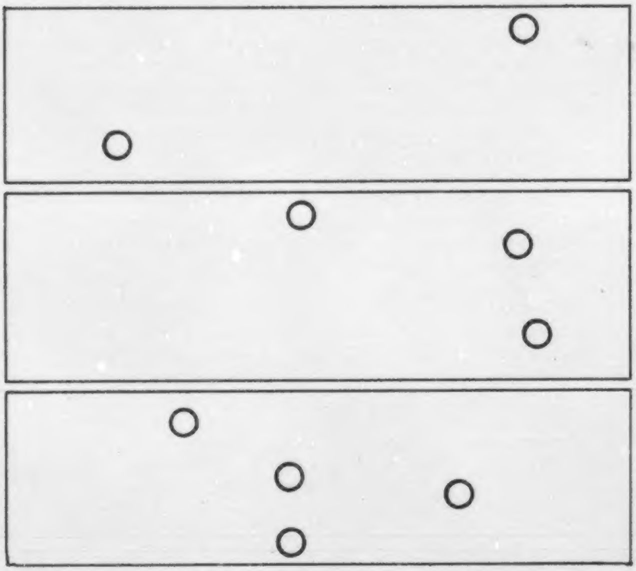


Figure 7. A stimulus containing a pair, a trio, and a quartet of dots which is used to study discriminative cardinality.

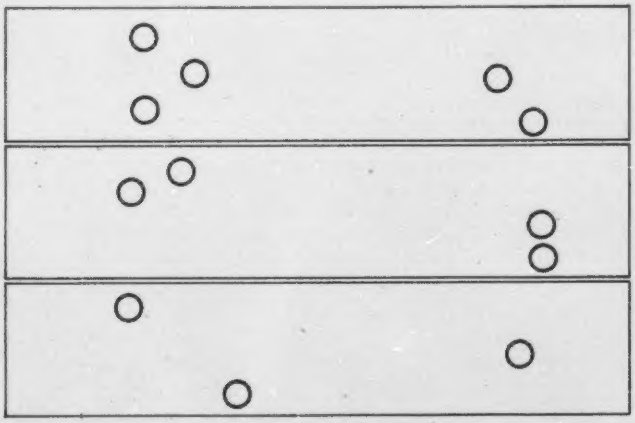


Figure 8. A stimulus containing a trio, a quartet, and a quintet of dots which is used to study discriminative cardinality.

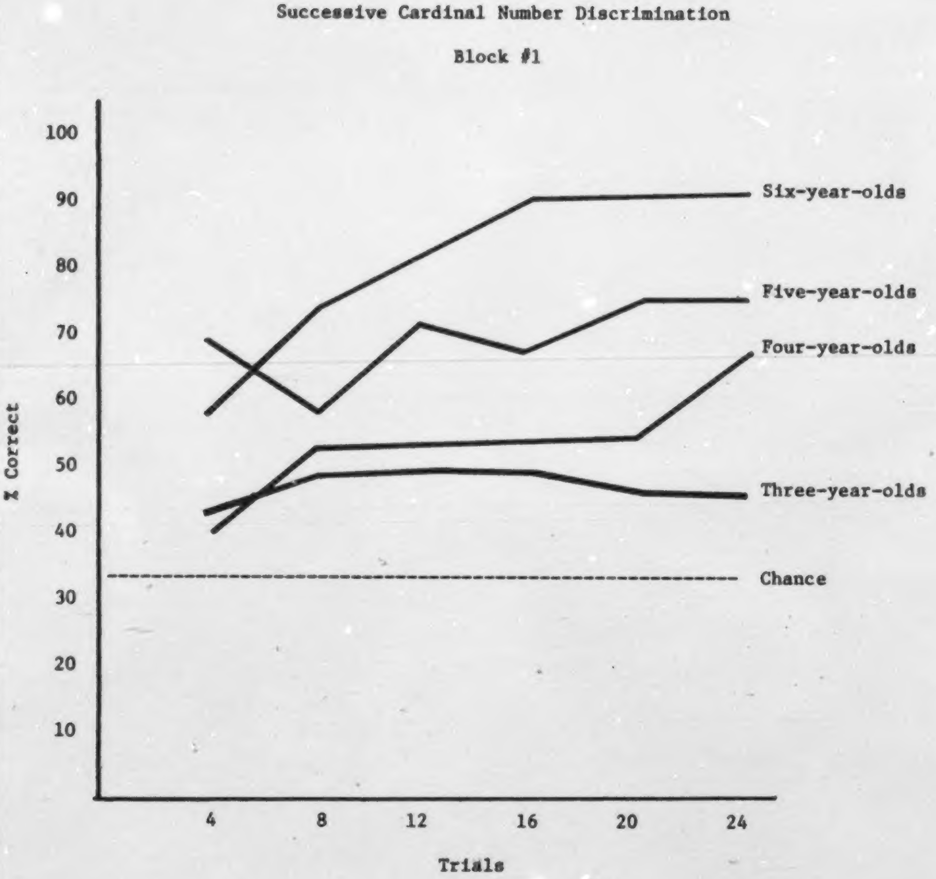
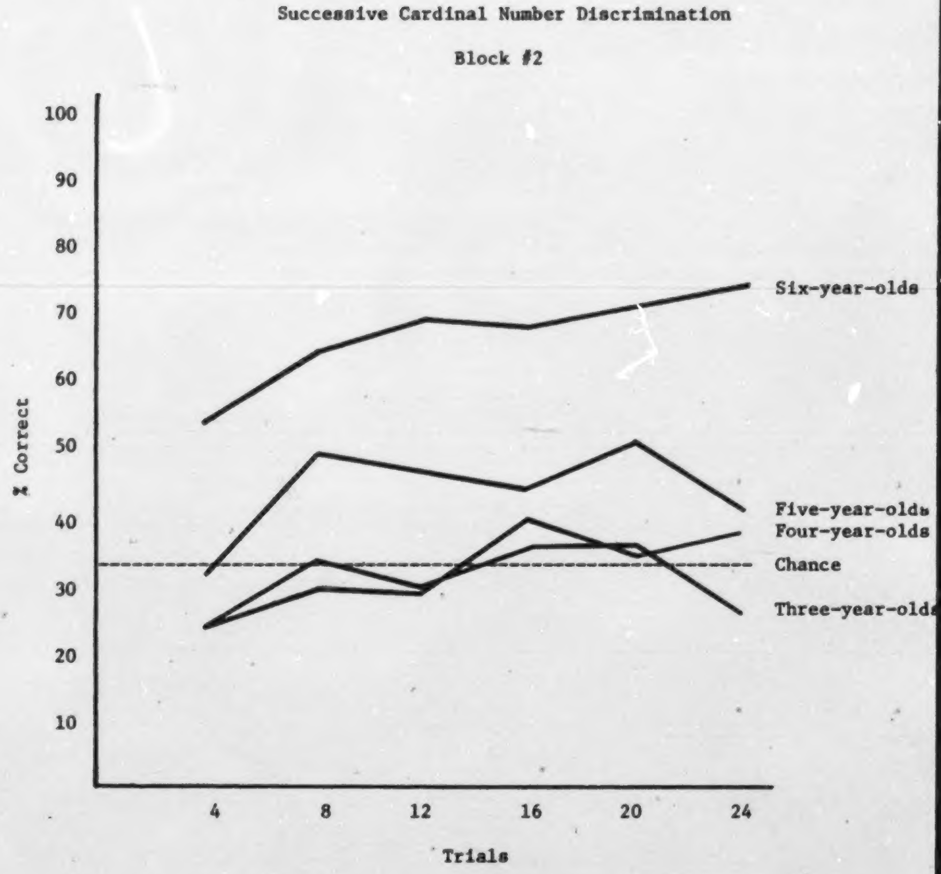


Figure 9. Average numbers of quartet responses made by three- to six-year-olds on stimuli resembling Figure 7. N = 35 per age level.


Successive Cardinal Number Discrimination

Block #2

Figure 10. Average numbers of quartet responses made by three- to six-year-olds on stimuli resembling Figure 8 after prior trials with stimuli resembling Figure 7. N = 35 per age level.

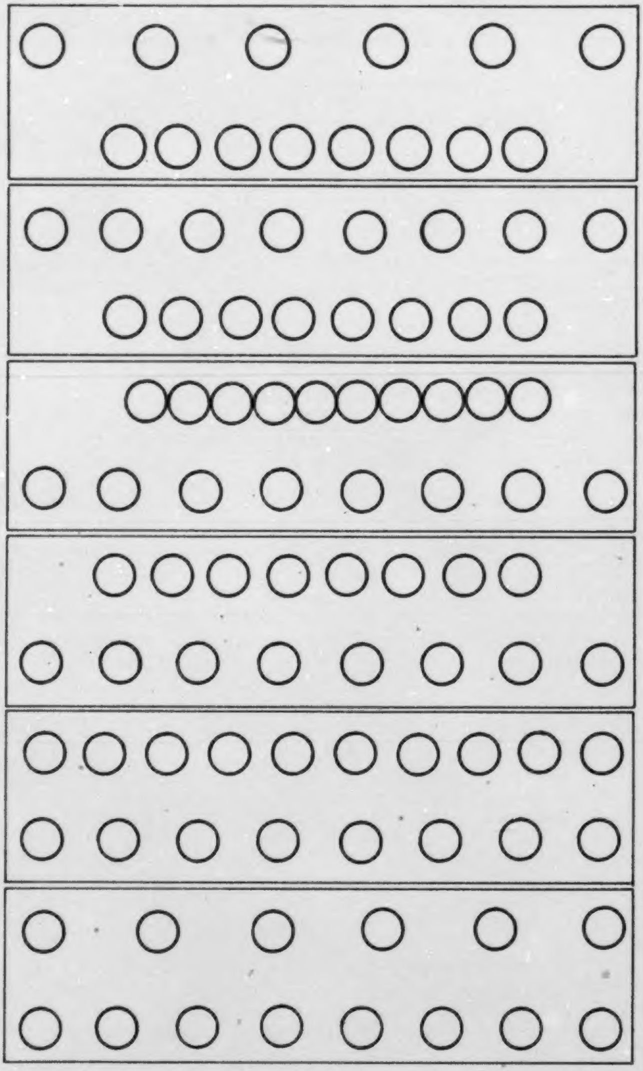


Figure 11. Six illustrative stimuli used to assess cardination.

The ordinal theory proposes that children do not possess complete internal cardinality, as indexed by the solution of cardination problems, before the second half of their elementary school years--and perhaps somewhat later.

Development of Ordinal, Cardinal, and Natural Number

The most distinctive feature of the ordinal theory of number development is a posited sequence in the development of numerical concepts which conflicts with certain beliefs which currently are popular in North American educational circles. The posited sequence is: ordinal number → natural number → cardinal number. It will be recalled that we are taking the initial facts of arithmetic, the capacity to manipulate the first few integers in prescribed ways, as our index of natural number. Therefore, the theory predicts that most children will make considerable progress in the ordinal sphere before they make any progress in either the arithmetic or cardinal spheres. Further, the theory predicts that children will acquire considerable facility with arithmetic before they grasp cardinal number. Historically speaking, the second and third predictions are the most important. Concerning the ordinal number → cardinal number sequence, it is usually assumed today that, cognitively, cardinal number either is a more basic concept than ordinal number or that the two are equally basic (Brainerd, 1975a, chap. 6). However, the ordinal theory maintains that most children grasp ordinal number long before they comprehend cardinal number. The theory predicts, more particularly, that each of the various levels of cardinal sophistication described above lags behind the corresponding level of ordinal sophistication. Concerning the natural number → cardination sequence, it is widely assumed, especially among mathematics educators, that the concept of cardinal number is more basic than arithmetic (Brainerd, 1975a, chaps. 6 & 11). More than any other factor, the advent of the so-called "new school mathematics" in North America has fostered this belief. However, the ordinal theory maintains that arithmetic initially is more basic than cardinal number.

Functionally, the ordinal theory maintains that the early growth of arithmetic is directly dependent on prior achievements in the ordinal sphere. Early arithmetic growth is presumed to be especially dependent on understanding the ordinal meanings of the first few natural numbers and the primary operations of arithmetic. Conversely, the ordinal theory proposes that the early growth of arithmetic is largely independent of prior achievements in the cardinal sphere. Although cardinal concepts may become relevant later on, the initial arithmetic skills do not appear to depend on either understanding the cardinal meanings of the first few integers or the cardinal meanings of the primary operations of arithmetic.

Empirical Evidence

Let us turn now to the literature on children's prearithmetic numerical concepts for purposes of determining the extent to which this literature confirms or disconfirms the predictions we have just considered. As I mentioned earlier on, I do not propose to review this literature exhaustively. In fact, I propose to restrict the review to three large-scale normative studies and two training experiments conducted at the University of Alberta and a replication study conducted under the auspices of the University of Wisconsin's Research and Development Center for Cognitive Learning. The former group of investigations was conducted between 1972 and 1974, while the latter study was conducted during late-1973 and early-1974. I justify reviewing the literature in this admittedly selective manner on three principal grounds. In the first place, as I observed previously, the great preponderance of currently available studies has nothing to do with our central theme, viz., developmental and functional connections between ordinal, cardinal, and natural number concepts. This is not to say that the literature is entirely devoid of investigations which provide evidence about the predictions with which we are concerned. In particular, studies reported by Beard (1963), Dodwell (1960, 1961, 1962), Hood (1962), Siegel (1971a, 1971b, 1974), Wang, Resnick, and Boozer (1972), and, of course, Piaget's classic studies in The Child's Conception of Number (Piaget, 1952) all provide pertinent evidence. However, and this is my second reason for restricting this paper to the aforementioned investigations, I have reviewed these other studies in detail on two previous occasions. These reviews may be found in part 3 of an earlier monograph (Brainerd, 1973a) and in chap. 7 of The Origins of the Number Concept. Third, each of the studies to which this review is confined was very expressly designed to get at one or more of the predicted relationships between ordinal, cardinal, and natural number that we considered above. This is definitely not true of other studies. On the whole, these latter studies were designed for quite different purposes and, consequently, such evidence as they provide vis-a-vis our central theme tends to be suggestive rather than conclusive.

Returning to the account of number development considered in the preceding section, virtually all of the findings that I am about to discuss are concerned with the final steps of this account. That is, they focus primarily on the relationships between the cognitive counterpart of ordinal number (ordination), the cognitive counterpart of cardinal number (cardination), and first arithmetic skills. Therefore, most of the subjects in these studies are children in the first few elementary grades. Taken together, these two facts entail that, unfortunately, we know very little at present about the early growth of ordinal and cardinal number. More to the point, we know very little about (a) the relationship between perceived ordinality and discriminative ordinality, (b) the relationship between perceived cardinality and discriminative cardinality, (c) the relationship between perceived ordinality and perceived cardinality, and (d) the relationship between discriminative ordinality and discriminative cardinality. Despite our comparative ignorance about ordinal and cardinal development during infancy and the preschool years, we now know a good deal about the final phases of number development in the ordinal, cardinal, and arithmetic areas. For the most part, what we know is consistent with the predictions of the ordinal theory. However, in view of the paucity of data on the earlier phases of cardinal and ordinal development, there is absolutely no empirical reason for supposing that the theory's proposals about the beginnings of number development are correct.

The first study (Brainerd, 1973a, 1973b) was a simple normative investigation conducted during early-1972. It was designed to focus narrowly on the developmental relationship between the cognitive counterpart of ordinal number and the cognitive counterpart of cardinal number. The ordinal theory, it will be recalled, predicts that the concept of ordinal number is present in most children by the time they enter elementary school whereas the concept of cardinal number is not present until considerably later. Hence, tests of ordination and cardination such as those described in the preceding section were administered to a sample of 180 kindergarteners and first-graders selected at random from several elementary schools located in middle-class areas of Edmonton. There were three general findings. First, there were three levels of performance on the ordination tests. Some children (Level I), always the youngest ones, appeared to be completely incapable of making a transitive inference; other children (Level II), somewhat older than those in the first group, could make transitive inferences in some situation but not in others; still other children (Level III), somewhat older than those in the second group, always made transitive inferences. The third group predominated. A clear majority of even the five-year-olds invariably made transitive inferences in all situations. The second major finding was that there also were three distinct levels of cardination performance. Some children (Level I), the youngest, based their relative manyness judgments on the relative length of the parallel rows of red and blue dots; other children (Level II), slightly older, based their relative manyness judgments on the relative density of the two rows; a final group of children (Level III), the oldest, based their relative manyness judgments on the type of correspondence between the two rows. The first group predominated: 54% of the sample based their relative manyness judgments on relative length; 37% of the sample based their relative manyness judgments on relative density; only 9% of the sample based their relative manyness judgments on term-by-term correspondence. The third and by far the most important finding of the study was that the ordinal number concept appeared to emerge in children's thinking long before the cardinal number concept. Slightly more than half of the children in this sample (93 of 180) had attained the highest level of ordination. Only 15 of these children had attained the highest level of cardination. Moreover, although only 12 of the 93 children who clearly possessed the concept of ordinal number also possessed the concept of cardinal number, 72 of the 92 subjects who were functioning at the lowest level of cardinal number clearly possessed the concept of ordinal number. The findings of this study appear in Table 1.

Table 1

The Developmental Relationship Between Ordination and Cardination

	Level of Cardination		
Level of Ordination	No correspondence	No one-one correspondence	Complete internal correspondence
No ordering	20	4	0
Partial ordering	32	17	3
Complete internal ordering	40	41	12

Note. Only 169 subjects who did not count during the cardination tests appear.

The second study (Brainerd, 1973a, 1973b) was similar in principle to the first. It was conducted during late-1972. A new sample of 180 kindergarten and first-grade children selected from middle-class Edmonton schools was administered the same ordination and cardination tests as in the above study. However, this new sample of children also was administered an arithmetic test. The test measured the children's understanding of the first 16 addition facts (i.e., 1 + 1 = 2, 1 + 2 = 3, ... , 4 + 4 = 8) and their understanding of the first 16 subtraction facts (i.e., 2 - 1 = 1, 3 - 1 = 2, ... , 8 - 4 = 4). The general aims of this study were, first, to determine whether the findings of the preceding study could be replicated and, second, to evaluate the developmental relationships between ordinal number and arithmetic, on the one hand, and cardinal number and arithmetic, on the other. Concerning the former aim, all three of the principal findings of the first study also were observed with the new sample of subjects. Three distinct levels of ordinal number performance were noted; three distinct levels of cardinal number performance were noted; and the concept of ordinal number was observed to be far in advance of the concept of cardinal number. Concerning the second aim, a classification scheme was developed for the arithmetic test to facilitate comparison with the ordinal and cardinal number tests. The arithmetic classification scheme consisted of three levels (below average, average, above average). It was developed through consultations with principals and teachers, and it was validated against age. When the children's ordinal number performance was compared with their performance on the arithmetic test, there was clear evidence that, in line with the ordinal theory's prediction, the ordinal number concept precedes arithmetic in children's thinking. A total of 119 children clearly possessed the concept of ordinal number, but less than half of these children also had attained the highest of the three levels of arithmetic proficiency. Importantly, roughly a third of these 119 children were still below average in arithmetic proficiency. When the children's cardinal number performance was compared with their arithmetic performance, the findings were the reverse of those just mentioned for ordinal number. A total of 95 children were functioning at the lowest of the three levels of cardinal number (i.e., longer = greater manyness). Roughly half of these 95 children evidenced either above average or average arithmetic proficiency. The findings of this study appear by type of test in Table 2.

It is obvious that, on the whole, the findings of the preceding two studies tend to confirm the ordinal number → natural number → cardinal number sequence posited in the ordinal theory. Before moving on to new findings, I should like to discuss another normative study concerned with this same sequence which recently has come to my attention. The study is, in most respects, a straight forward replication of the second of the two studies we have just examined. The study was conducted during late-1973 and early-1974 by a group of investigators headed by F. H. Hooper at the University of Wisconsin's Research and Development Center for Cognitive Learning. The Hooper group conducted the study as part of a six year longitudinal investigation of various Piagetian concrete-operational concepts. Although the data are as yet unpublished, Dr. Hooper generously passed along the findings of the replication in June of 1974. The motivation for this replication, at least as I understand it, is that the ordinal number → natural number → cardinal number sequence, though consistent with the predictions of the ordinal theory, is blatantly inconsistent with certain predictions of Piagetian theory. Piaget, for

Table 2

The Developmental Relationships Between Ordination, Cardination and Arithmetic Competence

Arithmetic	Ordination and Cardination					
	Level of Ordination			Level of Cardination		
	I	II	III	I	II	III
Addition proficiency						
Below average	20	17	35	55	10	0
Average	2	16	38	30	14	7
Above average	0	6	46	25	19	20
Subtraction proficiency						
Below average	7	14	16	18	3	0
Average	0	6	17	8	16	6
Above average	0	3	27	10	21	8

Note. The upper half of the table includes all 180 subjects who participated in Study II, but the lowest half of the table includes only the 90 first graders.

obscure reasons which I confess that I have never been able to fathom, maintains that developmental changes in the ordinal, cardinal, and natural number spheres always occur in tight synchrony with each other (e.g., cf. Beth & Piaget, 1966; Piaget, 1952, part 2). The findings of the two studies just discussed, whatever else they may show, certainly do not support the synchrony predicted by Piagetian theory. Therefore, to investigators who, like the Hooper group at the University of Wisconsin, are chiefly concerned with the verifiability of Piagetian theory, the replicability of the aforementioned findings is a very important issue indeed. In the study in question, the ordinal number, cardinal number, and arithmetic tests administered to the 180 children in the second of the preceeding studies were administered to a new sample of elementary schoolers. The new sample consisted of 50 kindergarten children and 50 third-grade children selected from elementary schools located in Beloit, Wisconsin. After all 100 children had been administered the three types of tests, the classification schemes mentioned earlier were used to assign each subject to (a) one and only one of the preceding three levels of ordinal number, (b) one and only one of the preceding three levels of cardinal number, and (c) one and only one of the preceding three levels of arithmetic proficiency. The results of the Wisconsin replication, although, as I said, still unpublished, recently have been made available in technical report form by one of the investigators in the Hooper group (Gonchar, 1974). I report these findings in Table 3. Note that, in so far as the ordinal number → natural number → cardinal number sequence is concerned, this sequence is every bit as apparent in Wisconsin children as in Edmonton children.

Table 3

The Developmental Relationship Between Ordination and Cardination for the Wisconsin Sample

Ordination Level	Cardination Level		
	I	II	III
Kindergarten			
I	9	0	0
II	18	3	0
III	23	7	0
Third-grade			
I	1	0	0
II	0	11	3
III	6	26	13

The fourth study (Brainerd, 1973b; Brainerd, 1975a, chap. 9) I should like to report, like the second study, was conducted during late-1972. Unlike the three studies considered up to this point, it was concerned with functional rather than developmental relationships. It will be recalled that the ordinal theory predicts that the emergence of first arithmetic skills is directly dependent on a prior understanding of ordinal number and is not directly dependent on a prior understanding of cardinal number. This leads to the general expectation that experimentally produced improvements in children's ordinal number concepts should lead to correlated improvements in arithmetic which are more pronounced than correlated improvements in arithmetic produced by experimentally induced improvements in cardinal number. The ordinal number → cardinal number sequence also leads one to expect that it will be more difficult to induce improvements in cardinal number in the laboratory than to induce improvements in ordinal number. To test these predictions, a training experiment was conducted in which some subjects received ordinal number training, some subjects received cardinal number training, and transfer to arithmetic proficiency was assessed. The experiment consisted of 10 different sessions spaced at intervals of one week. Each of the subjects who participated in the entire experiment was seen once per week on 10 separate occasions. During the first of the 10 sessions, 240 kindergarten children were administered the tests for ordinal number, cardinal number, and arithmetic employed in the aforementioned normative studies. Following the administration of these pretests, the children were divided into four groups of 60 subjects each in such a manner that the average levels of performance on the ordinal number, cardinal number, and arithmetic measures were exactly the same for all four groups. Two of the four groups were training conditions and the other two groups were control conditions. The subjects in one of the two training conditions received ordinal number instruction during each of the eight sessions immediately following the pretest session. During each of these sessions, the training manipulation consisted of presenting a graded series of ordination problems--seriation and transitive inference--involving such everday transitive-asymmetrical relations as "taller than, "larger than," etc. A simple correction training procedure was employed. That is, on the graded series of problems administered during each ordinal number training session, the subjects were corrected whenever they made an incorrect response, and they received a reward whenever they made a correct response. The graded series of ordinal number problems was administered twice during each training session. The subjects in the other training condition received cardinal number instruction during each of the eight sessions immediately following the pretest session. During these eight sessions, the training manipulation consisted of presenting a graded series of cardination problems. The easiest problems in the series involved making relative manyness judgments about collections containing a pair, a quartet, and a sextet of objects. The most difficult problems in the series were like the cardination pretest--i.e., they involved comparing classes which contained a sextet, an octet, and a decade of objects. As was the case in the ordinal number training condition, a simple correction procedure was employed. The subjects were corrected whenever they made an erroneous relative manyness judgment, and they were rewarded whenever they made a correct relative manyness judgment. The subjects in one of the two remaining conditions served as controls for the ordinal number training condition. During each of the eight sessions following the administration of the pretest, these subjects received the same graded series of ordinal number problems as the subjects in the ordinal number training condition. However, the correction procedure was omitted; the problems were simply administered without comment by the experimenter. The 60 subjects in the last condition served as controls for the 60 subjects in the cardinal number training condition. These subjects were administered the same graded series of cardinal number problems during the training trails as the cardinally trained subjects received; however, the correction procedure was ommited. After the eight training sessions had been completed, all 240 subjects received a readministration of the ordinal number, cardinal number, and arithmetic tests administered during the first session 10 weeks earlier. This was done to determine how the intervening training experiences had altered the children's grasp of ordinal number, cardinal number, and the initial facts of arithmetic.

A comparison of the subjects' performance on the tests administered during the first session with their performance on the same tests 10 weeks later revealed three major findings. First, it was noted that children's understanding of both ordinal and cardinal number predictably improved as a function of training. The average posttraining ordinal number performance of the ordinally trained subjects was significantly better than the average posttraining ordinal number performance of the ordinal controls. The average posttraining cardinal number performance of the cardinally trained subjects was significantly better than the average posttraining cardinal performance of the cardinal controls. The second major finding was that, consistent with the ordinal number → cardinal number sequence predicted by the ordinal theory, the training procedure induced more substantial improvements in ordinal number performance than it induced in cardinal number performance. Explicitly, the average pretest to posttest improvement in the ordinal number performance of the ordinally trained subjects was 30% greater than the average pretest to posttest improvement in the cardinal number performance of the cardinally trained subjects. The third major finding of this experiment was that ordinal number training tended to transfer better to the arithmetic area than cardinal number training did. The arithmetic performance of the ordinally trained subjects improved roughly 25% over the 10 week interval. When this value was corrected for the amount of spontaneous improvement in the ordinal controls, the correlated improvement in arithmetic as a function of ordinal number training turned out to be statistically significant. On the other hand, the arithmetic performance of the cardinally trained subjects improved roughly 12% over the 10 week interval. When this value was corrected for the amount of spontaneous improvement in the cardinal controls, the residual improvement in arithmetic as a function of cardinal training was not statistically significant. This latter finding is not necessarily predicted by the ordinal theory. All the ordinal theory predicts is that, independent of the absolute improvement in arithmetic which accrues as a result of ordinal and cardinal training, the improvements in arithmetic which result from ordinal training will be greater than those which result from cardinal training. The data for the first session and the tenth session appear by type of test and experimental condition in Table 4.

Table 4

Average Pretest and Posttest Scores of the Subjects Participating in Experiment I

Type of Test	Condition			
	Ordination training	Ordination control	Cardination training	Cardination control
Pretest				
Ordination[a]	5.64	5.64	5.64	5.64
Cardination[b]	0.96	0.96	0.96	0.96
Arithmetic[c]	5.12	5.12	5.12	5.12
Posttests				
Ordination[a]	11.38	6.60	6.24	6.13
Cardination[b]	1.32	1.19	2.76	1.56
Arithmetic[c]	14.13	8.32	8.96	7.88

[a] High possible score for each cell = 12.

[b] High possible score for each cell = 12.

[c] High possible score for each cell = 32.

The next study which I should like to discuss (Brainerd, 1974) was undertaken in the hope of providing an explanation of the training experiment that we have just considered. The findings of the preceding training experiment, especially those on the relative transfer of ordinal number and cardinal number training to arithmetic, tend to support the ordinal theory's claim that the emergence of first arithmetic skills is more dependent on a prior grasp of ordinal number than on a prior grasp of cardinal number. But why should knowledge of ordinal number have a greater impact on arithmetic proficiency than knowledge of cardinal number? The following explanation suggests itself. Arithmetic proficiency, at least in the beginning, consists simply in being able to combine the numerals "1," "2," "3," and so forth in certain prescribed ways. Now, I should say that the numeral symbols whose combination forms the subject matter of arithmetic are quite abstract and bear no obvious physical resemblance to anything in our everyday environments. Speaking as a psychologist, I should think that it would be considerably easier for children to learn to manipulate these symbols in appropriate ways if the individual symbols did not remain entirely abstract but, rather, were assigned concrete meanings of some sort. We know that each of these symbols has two such meanings--one a positional meaning and the other a manyness meaning. Suppose, just for the sake of argument, that it is much easier for children to learn to associate each of the numerals of arithmetic with its correct ordinal meaning than it is to associate that same numeral with its correct cardinal meaning. In other words, suppose that it is easier to associate the numeral 1 with "first" than with "singleton," that it is easier to associate the numeral 2 with "second" than with "pair," that it is easier to associate the numeral 3 with "third" than with "trio," and so on. If this happened to be true, then we would have a possible explanation of the fact that ordinal number training produces greater correlated improvements in arithmetic than cardinal number training. The first form of training, after all, provides a means for acquiring the easier ordinal meanings of numerals, whereas the latter form of training provides a means for acquiring the considerably more difficult cardinal meanings of numerals. The experiment whose procedure and findings I shall now summarize was designed to test this line of reasoning.

The general aims of the experiment were, first, to obtain a sample of children who did not know the ordinal and cardinal meanings of the first five numerals and, second, to train the children to acquire these meanings. The experiment consisted of three sessions. During the first session, 159 preschoolers (average age = 4 years, 7 months) were administered three types of pretests: (a) numeral identification, (b) ordinal numeral meaning, and (c) cardinal numeral meaning. On the numeral identification pretest, the subjects had to recognize the name of the numerals 1, 2, 3, 4, and 5. The ordinal numeral meaning pretest was designed to determine whether the subjects knew the correct positional meanings of these same five numerals. Stimuli like the one shown in Figure 12 were used on the ordinal numeral meaning pretest. The experimenter would point in turn to each of the five symbols in the box at the top of Figure 12 and ask the subject to select the term in the progression at the bottom of the figure which went with that particular numeral. The cardinal numeral meaning pretest was similar in that it was designed to determine whether the subjects knew the correct manyness meanings of the first five numerals. Stimuli like the one shown in Figure 13 were used on the cardinal meaning pretest. The experimenter would point in turn to each of the five symbols in the box at the top of Figure 13 and ask the subject to select the collection at the bottom of the figure which went with that particular numeral. After the pretests had been completed, 120 children were selected for the training phase who met three criteria: they could recognize and name each of the first five numerals; they did not know the ordinal meanings of these same numerals; they did not know the cardinal meanings of these same numerals.

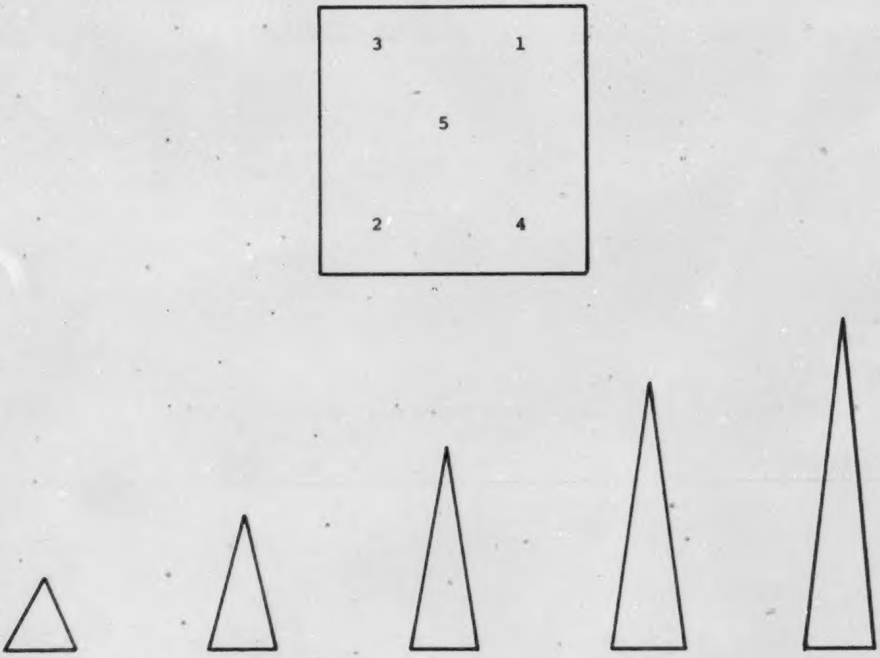


Figure 12. An illustrative stimulus used to assess preschoolers' grasp of the ordinal meanings of the first five numerals.

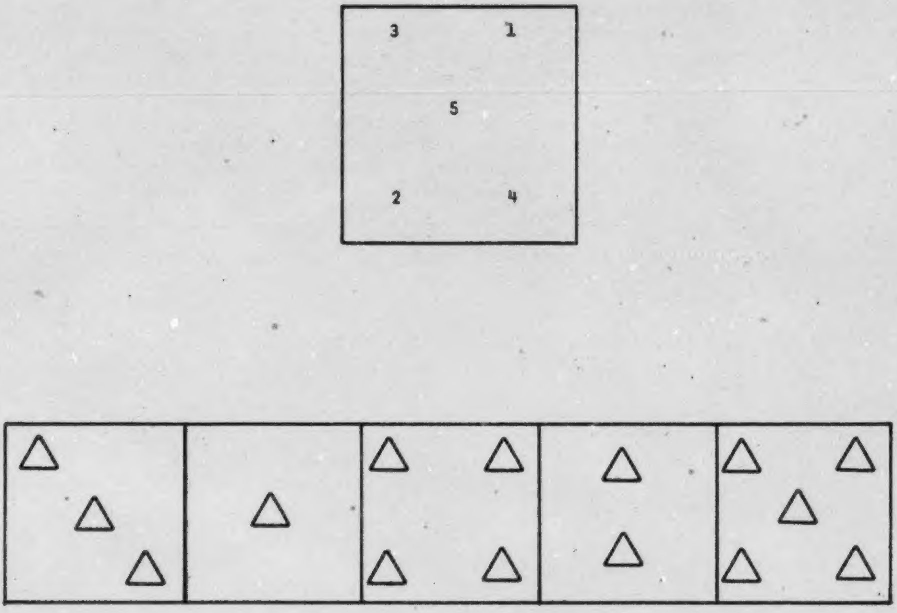


Figure 13. An illustrative stimulus used to assess preschoolers' grasp of the cardinal meanings of the first five numerals.

As was the case in the earlier training experiment, the remaining 120 subjects in this experiment were divided into four groups: ordinal numeral training, ordinal numeral control, cardinal numeral training, and cardinal numeral control. During the remainder of the first session, the subjects in each condition were given 12 training trials. One week later, during the second session, four more training trials and a series of posttests were administered. One week after that, during the third session, the series of posttests was readministered. The 16 training trials for the ordinal numeral training condition were designed to teach the children to associate each of the first five numerals with its correct positional meaning. A series of stimuli like the one in Figure 12 were presented on each of the training trials, and a simple correction procedure was instituted. Each time a subject made an incorrect positional judgment, he or she was corrected; each time a subject made a correct positional judgment, a reward was provided. The subjects in the ordinal numeral control condition were administered the same series of training trials, except that correction was omitted. The subjects in the cardinal numeral training condition were given 16 training trials designed to teach them to associate each of the first five numerals with its correct manyness meaning. A series of stimuli like the one in Figure 13 were presented on each of the cardinal numeral training trials, and the same correction procedure used to train the ordinal meanings of numerals was instituted. The subjects in the cardinal numeral control condition were administered the same series of training trials, except that correction was omitted. After the four training trials given at the beginning of the second session had been administered, the subjects in all four conditions received three ordinal meaning posttests and three cardinal meaning posttests. One week later, all six posttests were repeated.

The findings of the experiment I have just described generally tended to show that, consistent with the hypothesis proposed earlier, it is easier for children to learn the positional meanings of numerals than it is for them to learn their corresponding manyness meanings. On both the immediate (second session) and delayed (third session) posttests, the grasp of positional meaning evidenced by the children in the ordinal training condition was far better than the grasp of manyness meaning evidenced by the children in the cardinal training condition. On the three ordinal posttests administered during the second session, the children in the ordinal training condition selected the correct numeral for a given position 82.9% of the time. When the same ordinal posttests were administered one week later, the ordinally trained subjects selected the correct numeral 72.4% of the time. The corresponding percentages for the cardinally trained subjects were considerably lower. On the three cardinal posttests administered during the second session, the children in the cardinal training condition selected the correct numeral for a given manyness 61.8% of the time. When the same three cardinal posttests were administered one week later, the cardinally trained subjects selected the correct numeral 45.6% of the time. In addition to being easier to learn in the first place, the data of this experiment indicated that the ordinal meanings of numerals also are retained better. Across the one week interval between the immediate and delayed posttests, the performance of the ordinally trained subjects declined 10.5%. During the same time interval, the performance of the cardinally trained subjects declined 16.2%. It turns out that these two rates of long-term retention are significantly different statistically ($p < .01$). The complete results of this experiment appear in Table 5. Please note that the high possible value for each cell in this table is 10.00.

The final study which I shall consider (Brainerd, 1975a, chap. 10) was concerned exclusively with the growth of cardination and other cardinal number ideas during the elementary school years. It will be recalled that we are defining "cardination" as knowing that the relative manyness

Table 5

Average Numbers of Correct Numeral Choices for the Four Conditions

Test	Condition			
	Ordinal training	Ordinal control	Cardinal training	Cardinal control
Pretests (Session 1):				
Ordinal	2.23	2.00	2.40	2.47
Cardinal	2.37	2.13	1.83	2.17
Pretests (Session 2):				
First				
Ordinal	8.93	3.47	2.17	2.33
Cardinal	1.93	2.37	6.87	2.27
Second				
Ordinal	8.77	3.27	2.23	2.47
Cardinal	1.87	2.13	6.63	2.17
Third				
Ordinal	7.17	2.73	2.27	2.43
Cardinal	2.23	2.23	5.03	2.57
Posttests (Session 3):				
First				
Ordinal	8.53	2.93	2.27	2.47
Cardinal	2.07	1.83	5.17	2.23
Second				
Ordinal	8.07	2.53	2.33	2.53
Cardinal	2.13	2.23	5.07	2.67
Third				
Ordinal	5.13	2.63	2.47	2.17
Cardinal	2.07	2.43	3.33	2.53

Note. For each cell, 10 is the highest possible number of correct choices.

of two collections is completely determined by the type of correspondence that obtains between them. It will also be recalled that the ordinal theory predicts a protracted period of emergence for cardinal ideas spanning at least the middle-childhood years and perhaps continuing on into adolescence. To test this prediction, we administered the cardination tests described earlier to 350 children between the ages of five and 11. A total of 50 children was selected from each of the grades kindergarten through six. In addition to cardination tests, these children received tests for five other cardinal concepts.

Four of the additional concepts were believed, primarily on logical grounds, to be necessary preconditions for cardination. These concepts were class intension, conservation of cardinal equivalence, class extension, and multiple classification by manyness. Each of these concepts should be a necessary precondition for cardination because while each of them is concerned with classes and/or manyness and/or correspondence, none of them involves understanding the connection between type of correspondence and relative manyness as cardination does. Piaget's familiar object sorting task (Inhelder & Piaget, 1964) was the test for the class intension concept. Piaget's so-called "number" conservation problem (Piaget, 1952) was the test for conservation of cardinal equivalence. The test for class extension was a card sorting task in which there were eight cards that varied on three binary dimensions, viz., color (yellow or black), form (square or triangle), and cardinal number (sextet or octet). The aim was to determine whether the subject could sort the cards according to relative manyness as well as according to color and form. The test for multiple classification by manyness was a variation of Piaget's matrix problem (Inhelder & Piaget, 1964). The problem consisted of 2 x 2 matrices in which either the rows or the columns varied in terms of their cardinal number (sextet or octet). In addition to the four cardinal concepts just described, a fifth concept was measured which was believed to presuppose a prior understanding of cardination. This concept was the class inclusion principle. Class inclusion, like cardination, would seem to presuppose an understanding of the connection between type of correspondence and relative manyness. However, whereas cardination involves understanding this connection for two physically distinct classes, class inclusion involves understanding it for two classes which are not physically distinct--i.e., a subordinate class and a superordinate class. Logic suggests that applying one's knowledge of the correspondence-relative manyness relationship in the latter situation would be more difficult than in the former situation.

All of the tests just mentioned were administered to each of the 350 subjects participating in the study. As we saw above, a three level classification scheme already existed for performance on the cardination tests. To facilitate comparison of children's cardination performance with their performance on the other tests, identical three-level schemes were constructed and validated against age for each of the other five tests. After all of the children had been assigned to one and only one of the three levels of each concept, their relative levels of performance were examined. There were three major findings. First, of all of the six concepts studied, only the concept of class intension was understood by virtually all children at the time they entered elementary school. The other five concepts tended to evolve gradually throughout the elementary school age range. Interestingly, the class intension concept is the only one of the six that has nothing to do with either manyness or correspondence. Second, the concepts emerged in a clear developmental order. The subjects understood class intension first, conservation of cardinal equivalence second, class extension third, multiple classification by manyness fourth, cardination fifth, and class inclusion sixth. The third finding was the most important of all. The concepts of cardination and class inclusion, which are the crucial concepts in the study because both presuppose grasping the correspondence-relative manyness relationship, were never understood by more than a minority of the children. Even at the oldest age level, 11-years-old, slightly less than half the subjects clearly understood cardination, and only about one-third clearly understood class inclusion. The findings of this study are summarized in Figure 14. In this figure, the findings are reported by age level of subjects, type of concept, and level of performance. All the points on each curve are means for given age groups.

The data in Figure 14 obviously are consistent with the protracted period of cardinal number development posited by the ordinal theory. Insofar as knowledge of the correspondence-relative manyness relationship, in particular, is concerned, the data indicate that the evolution of this notion continues into adolescence. Given that this study did not include adolescents, it is impossible to say precisely when children may be expected to understand this relationship. However, some recent findings by F. H. Hooper and his co-workers at the University of Wisconsin shed some light on this problem. Hooper has found that the evolution of the class inclusion principle probably is not complete until the second half of the high school years.

What the Future Holds

Predicting future research directions in a given content area and offering recommendations about what sorts of studies investigators ought to contemplate are hazardous occupations at best. However, I think that the data we have just reviewed entail some fairly obvious suggestions about where we should go from here.

The first prediction-recommendation concerns the ordinal and cardinal skills of preschoolers. We now possess reasonably extensive evidence on the sophisticated cognitive counterparts of the logician's definitions of ordinal and cardinal number in elementary schoolers.

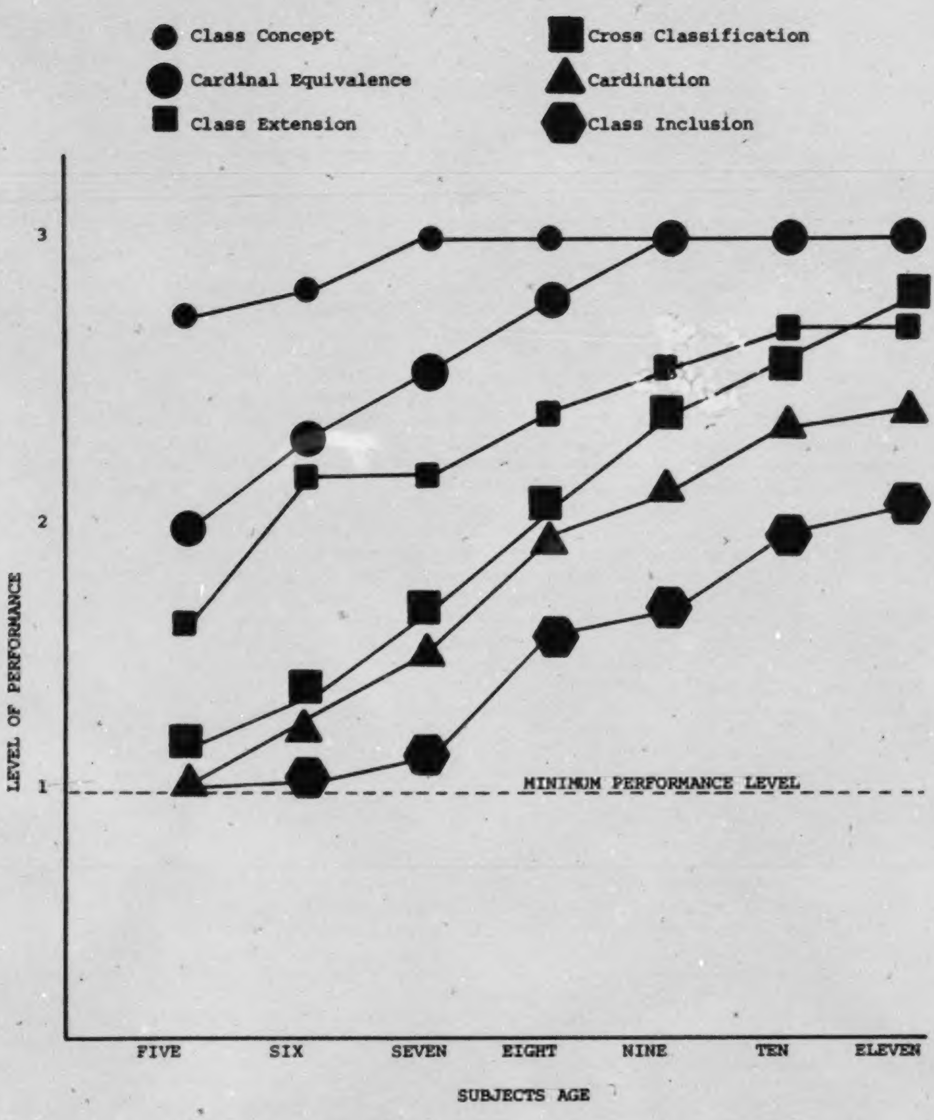


Figure 14. Age-related changes in six cardinal number concepts. N = 50 per age level.

These data tend to show that, at least during the elementary school years, ordinal number is developmentally prior to cardinal number, and ordinal number is more closely connected with the initial emergence of arithmetic. Importantly, replications of the studies which adduced these data either have been completed by investigators in other laboratories or are now in progress. But what about earlier age levels? In contrast with our rather extensive findings on elementary schoolers, we know very little about the preschool precursors of the sophisticated ordinal and cardinal concepts of middle-childhood. In particular, we know very little about age-related changes in discriminative ordinality and discriminative cardinality. The ordinal theory, as we already have seen, makes clear developmental predictions about these two antecendents of ordinal and cardinal number, but the data simply are too thin at present to say whether or not the predictions are correct. It is true that some studies of discriminative ordinality and discriminative cardinality are now under way in Linda Siegel's laboratories at McMaster University and in my own laboratories at the University of Alberta. However, this research is far from complete. To date, Siegel and I have studied discriminative ordinality and discriminative cardinality in the context of only one simple discrimination learning paradigm, viz., successive discriminative learning with correction. Clearly, much more comprehensive work needs to be done with other paradigms--e.g., simultaneous discrimination--before we can be certain of our facts. Therefore, I would like to suggest that, in the short term, we need to focus considerable attention on preschoolers' numerical ideas. We need to know, first, whether or not the age-related changes in numerical ideas which take place during the preschool years are consistent with what the ordinal theory predicts. If the data happened to confirm the predictions, we would still need to know precisely why the predictions are correct. Assuming that a plausible explanation could be found, either by appealing to task variables or to general laws of children's learning or to both, we would then have to confront the question of whether or not the explanation is consistent with what we know about preschoolers' discrimination learning in nonnumerical content areas. In brief, we have just scratched the surface of preschool numerical reasoning, and we need to know a very great deal more.

My second prediction-recommendation is the most tenuous of the three. It concerns the preconceptual skills of perceived ordinality and perceived cardinality during infancy. While we know very little about preschoolers' ordinal and cardinal concepts, we know nothing about the assumed perceptual harbingers of these concepts. As was the case for discriminative ordinality and discriminative cardinality, the ordinal theory makes definite developmental predictions about perceived ordinality and perceived cardinality. I know of no extant data which provide the slightest grounds for supposing these two predictions to be correct. Although perceived cardinality has occasionally been studied in preschoolers and elementary schoolers (cf. Nelson & Bartley, 1961, for some examples), perceived ordinality has never been studied, at least not that I know of, and, more important, perceived ordinality and perceived cardinality never have been compared using infant subjects. To evaluate the predictions of the ordinal theory vis-à-vis the perceptual roots of ordinal and cardinal number properly, we shall have to study infants. Of course, infants are not the easiest of subjects to work with. However, conditioning methods developed by Lipsett (e.g., 1969), Bower (e.g., 1974), and others are well-suited to the task of examining the ordinal theory's predictions in infant subjects. Thus, in addition to studying the immediate precursors of ordinal and cardinal number in preschoolers, I think we must seriously consider studying their more distant perceptual antecendents in infants. Our first step might be to use respondent and operant procedures to determine whether there is a developmental lag between perceived ordinality and perceived cardinality. Another extremely interesting question--one which mathematicians and philosophers as well as psychologists would very much like to have answered--that also might be investigated is when perceived ordinality and perceived cardinality first appear. There is an old and acrimonious debate among philosophers of mathematics which concerns whether or not there is an innate perceptual subtratum on which the human number concept is built (cf. Brainerd, 1975a, chap. 1). If it could be shown either that (a) perceived ordinality and/or perceived cardinality, like size and shape constancy, are present during the first weeks of life or that (b) neither appears until much later, then this ancient debate might at last be resolved.

My third and final prediction-recommendation is rather more down to earth than the two which have preceded it. It concerns the educational ramifications of the research reviewed in this paper. I have written at some length of the implications of number development research for education in chapter 11 of The Origins of the Number Concept, and I do not wish to rehash the arguments appearing therein at this point. Readers who wish a detailed treatment of the educational questions posed by number development research are directed to this chapter. Here, I wish only to note that a serious and sober reassessment of our current approach to defining arithmetic concepts in the public schools may be in order. As everyone knows, we have, for about a decade and one-half now, been emphasizing the meaning of arithmetic concepts as well as drill in the elementary classrooms of North America. Interestingly, and I should also say unfortunately, the method of teaching the meanings of arithmetic concepts that currently predominates is a purely cardinal one. In grade one, we initiate arithmetic instruction with the cardinal definitions of the first ten numerals. Later on in first-grade and in subsequent grades, children are taught cardinal definitions of arithmetic operations and other key concepts. Predicating arithmetic concepts entirely on cardinal ideas such as manyness, correspondence, etc. obviously presupposes that most children can and do grasp such ideas at an early age. This assumption is very explicitly stated by authors of books dealing with the philosophy of the new school mathematics (e.g., Johnson & Rahtz, 1966). But the research does not support the assumption. Quite to the contrary, if the evidence we have examined tends to show anything, it shows that cardinal ideas are very difficult for most children (middle-class children at that) to comprehend. Recall, for example, the normative data on the key concepts of cardination and class inclusion in the last of the studies which we reviewed. The cardinal definition of numerals, which we are now teaching to first-graders, involves understanding the connection between the type of correspondence which obtains between two classes and the relative manyness of their terms. But the normative data indicate that this notion is not generally understood until early adolescence. It is interesting to observe that all of the children who participated in this normative study were enrolled in schools employing cardinally-oriented arithmetic curricula.

Although I am admittedly a neophyte insofar as the rhetoric and politics of public school education are concerned, honesty and the data compel me to conclude that is it time to consider eliminating the theory of classes--or "set theory" as it is usually called in mathematics education--from our elementary arithmetic curricula. Our best evidence is that the definitions of arithmetic concepts which we currently expect our children to learn are far too difficult, and, consequently, they are worse than no definitions at all. This is not to say that we must give up trying to define arithmetic concepts and revert to pure drill. In place of cardinal definitions, I would strongly urge that we consider the merits of ordinal definitions. Although the data indicate that cardinal ideas are very difficult for elementary schoolers, they also indicate that ordinal ideas are comparatively easy for them. By the time they enter elementary school, most children appear to grasp all the key concepts that we would need to use to define numerals, arithmetic operations, and other concepts ordinally. Therefore, unlike cardinal definitions, ordinal definitions should not be especially difficult for children to learn.

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